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STUDIES IN THE α-PARTICLE MODEL OF NUCLEI

III. Calculation of the interaction between α-particles

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Received 11 October 1956

Abstract: The interaction between α-particles is investigated and compared with the requirements of the α-particle model. These are that there should be a short-ranged repulsion to prevent the α-particles from overlapping, and an additive attractive force reaching beyond the repulsion.

The interaction energy is divided into four components of different origin and properties. These components are a direct interaction (V_D), which is analogous to the Coulomb interaction between ions and which exists only if the nuclear potential has a direct (i.e. non-exchange) tail, a nucleon-exchange interaction (V_X), a polarization interaction (V_{POL}), and the Coulomb interaction E_C.

The nucleon exchange operator V_X is velocity dependent; its expectation value is calculated to the lowest order of overlap integrals with a wave function in which each of the interacting α-particles is confined to the neighbourhood of a fixed point. It is shown that to this approximation V_X is repulsive for central forces between nucleons, and it is assumed that the complete V_X is responsible for the repulsion required by the α-particle model. It is also shown that the tail of V_X is additive or not according to whether the force between nucleons does not or does have a direct component. The magnitude of the non-additive exchange energy which comes from the direct tail of the Serber potential is shown to be so small as to be not inconsistent with an α-α interaction which is additive for most purposes.

The direct interaction V_D and the polarization interaction V_{POL} are both additive attractive forces reaching out beyond the exchange repulsion. If the force between nucleons does not have a direct tail, then V_{POL} is the only attraction, but is too weak if the forces between nucleons are central. If, however, there is an attractive direct tail, then there is a direct attraction supplemented by V_{POL}, and it turns out that the strength of the direct tail in the Serber potential gives the right order of magnitude for the α-α attraction. The appeal to a direct component in the force between nucleons leads to the well-known difficulty with saturation, but arguments are given to show that this difficulty is not crippling.

It is shown that where the direct attraction predominates, the α-α interaction energy can be represented by a static potential; at shorter separations of the α-particles, V_X predominates and the α-α interaction can be represented by a potential only if one assumes that because of the strongly repulsive nature of V_X one may replace the latter by an effective repulsive potential. The complete interaction potential between α-particles which one obtains in this way agrees in order of magnitude but disagrees in shape with the parabolic potentials which have been used in published treatments of the energy spectra of some α-nuclei.

It is concluded that the attraction between α-particles required by the α-particle model can be accounted for if the force between nucleons has a direct component like that in the Serber potential. The mechanism of the attraction entails no more than a small probability of the α-particles' being dissolved in nuclei with $A \leqq 40$.

1. Introduction

The α-particle model pictures atomic nuclei as being made up mainly out of α-particles, each containing two neutrons and two protons, with possibly some loosely bound odd neutrons and protons. The α-particles are thought to move relative to one another much as do atoms in molecules. This paper discusses the derivation of the forces between the α-particles from the elementary forces between nucleons, and inquires whether the α-α interaction can have the following properties which are necessary if the α-particle model is to give a satisfactory description of nuclei:

(1) In order that the α-particles should retain their identity, and should not dissolve, the interaction forces must become strongly repulsive at α-α-spacings of less than about 3×10^{-13} cm where the α's start to overlap.

(2) To sustain nuclei against the electrostatic repulsion, the α-α forces must be attractive over α-α spacings of $3-4\times10^{-13}$ cm, corresponding to the average spacing of two α's in a nucleus [1]).

(3) In the region of the most probable α-α-spacings the α-α forces must be additive, i.e. the forces between a pair of α's must be independent of the presence of any other α's in nuclei consisting entirely of α-particles [1, 2]).

(4) The expectation value of the nuclear interaction energy between two neighbouring α-particles must be about 5 MeV in order to account for the binding energies of the α-nuclei [1]). ($A = 2N = 2Z = 4, 8, 12 \ldots 40$.)

There have been a number of discussions of the interaction between α-particles which have for the most part used the analogy with the interaction between atoms. Heisenberg [3]) proposed that the short range repulsion should be accounted for by nucleon-exchange forces, and that the additive attraction should be due to Van der Waals forces. Heisenberg remarked that the attraction was achieved by a polarisation of the α-particles, and therefore implied a tendency for them to dissolve. This tendency to solution presented a difficulty for the model. More extensive calculations were made by Margenau [4]), who claimed that the scheme proposed by Heisenberg could not produce additive forces because the additive Van der Waals attraction would have a shorter range than the first-order forces, which he supposed to be non-additive. (By "first order forces" is meant the nuclear analogue to the sum of electron exchange and Coulomb interactions between ions.)

An attempt to remove the difficulty about solution through polarisation was made by Inglis [5]). Inglis showed that if the interaction potential between two nucleons has a tail of Yukawa shape, rather than of the Gaussian shape which had been used by Heisenberg and Margenau, and if that tail contains an appreciable non-exchange part, then the first-order forces would have an attractive region. Inglis took the view that his first-order attraction was a consequence primarily of the long tail of

his nuclear potential, although the exchange character also had some importance.

The effect of a long tail in the nucleon-nucleon potential was further investigated by Edwards [6]). Edwards calculated the energy of two α-particles with a Yukawa potential, and also investigated the effect of the exchange character of the potential. He emphasised that the α-α attraction which Inglis had discovered was a consequence not of the shape of the potential, but of the fact that the tail did not contain any exchange operators. Edwards showed that the two – α system could be bound only if the nucleon-nucleon potential contained a sufficiently strong component depending only on the distance; this implied that if one wanted to bind Be^8 (which is actually just unstable) then one had to violate the saturation conditions. Because of the magnitude of the contribution of the nucleon exchange effect to the energy, Edwards concluded that the resultant attraction would not be additive.

To summarise the situation as it appears from the literature: it is possible to account qualitatively for an attraction, but only at the cost of either tending to dissolve the α-particles, or violating the saturation conditions; neither of these alternatives gives additive forces.

The analogy to the interaction of atoms on which all the calculations (except Edwards') were based contains an obvious weakness: there is nothing inside an α-particle to take the place of the massive nucleus in an atom; the adiabatic method familiar from the treatment of atomic interactions (see e.g. ref. 7) cannot therefore be used. This difficulty was overcome by Heisenberg [3]) by allowing the mass centres of the two interacting α-particles to oscillate with an arbitrarily chosen amplitude about two fixed points. Essentially the same device was used by Margenau [4]), except that the mass centre oscillation was described by the same parameter that described the motion of a nucleon inside an α-particle. By the use of this device, it was possible to obtain expressions for an interaction energy, but these expressions contained Heisenberg's mass centre oscillation parameter explicitly, and Margenau's implicitly, and are therefore difficult to interpret. Edwards [6]) overcame the difficulty by the alternative device of calculating the expectation value of the energy of two interacting α-particles with a trial wave function to describe the relative motion. This method has the disadvantage that it is difficult to investigate the distance dependence of the interaction energy.

In this paper, the interaction energy of α-particles will be calculated by new methods. We shall consider only nuclei which have $2N = 2Z = A = 4n$, where n is an integer. It was shown in an earlier paper [8]), which was based on some work due to Wheeler [9]), that if one assumes that the nucleons in a $4n$ nucleus are clustered into α-particles, then this system of α-particles

can be described by a wave function $F(\mathbf{X}_1 \ldots \mathbf{X}_n)$ which obeys the wave equation

$$-\frac{\hbar^2}{2M}\sum_{i=1}^{n}\nabla_i^2 F+(V_D+V_X+V_{POL})F = (E-nE_0^{(1)})F, \qquad (1.1)$$

where $\mathbf{X}_i$ $(i = 1, 2, \ldots n)$ are the co-ordinates of the mass centres of the α-particles,

M is the mass of an α-particle,

E is the total energy of the system,

and $E_0^{(1)}$ is the energy of an α-particle in the ground state.

V_D, V_X, and V_{POL} are operators which correspond to three qualitatively different types of interaction whose salient properties are summarised in table 1.1. Each of these operators has an anologue in molecular theory;

TABLE 1.1

Properties of the non-Coulomb parts of the interaction operators.

Interaction	*Spatial extent* (see ref. [8]))	*Additivity*	*Sign*	*Analogue in molecules* (see ref. [8]))
V_D (Direct interaction)	Is active for two α-particles when the separation of the mass-centres is less than about (diameter of α-particle + range of elementary nuclear force)	Additive	Repulsive or zero for saturating forces. Attractive for non-saturating forces (see § 2)	Long range Coulomb forces between ions.
V_X (Nucleon exchange interaction)	Appreciable only when two α's overlap	Mainly non-additive, but the tail can be additive	Repulsive (see § 3)	Heitler-London forces between rare gas atoms
V_{POL} (Polarisation interaction)	Intermediate between V_X and V_D	Additive except for a negligible component	Attractive (see § 4)	Van der Waals force

rules by which they can be calculated were given in ref. [8]) and will be further developed in this paper.

For the most part, ref. [8]) dealt with only two α-particles, but the generalisation to more than two is straightforward. V_D is the quantity defined in eqs. (3.27) and (7.2) of ref. [8]). For two α-particles, $V_X F$ is the term $\int d\boldsymbol{\xi}\, P(\mathbf{X}, \boldsymbol{\xi})F(\boldsymbol{\xi})$ given in eq. (3.30) of ref. [8]). The generalisation of V_X to more than two α-particles is given in § 3 of this paper.

Each of the three interaction operators comes partly from nuclear and partly from Coulomb forces. To simplify the presentation, we shall ignore the Coulomb forces until we come to compute the complete α-α interaction

in § 5, where we shall calculate the Coulomb energy with neglect of nucleon exchange and polarisation, and add it to the rest of the α-α interaction. This procedure is equivalent to consistently taking the Coulomb energy into account in V_D, and neglecting it in V_X and V_{POL}. The interactions denoted by V_D, V_X and V_{POL} in the rest of this paper are the nuclear force parts of the three operators in (1.1), while the Coulomb energy E_C which is added in § 5 is the Coulomb part of V_D in (1.1).

We shall give a quantitative discussion of the components of the α-α interaction. The discussion is based on the following assumptions:

(1) Nuclear matter in a $4n$ nucleus is clustered into α-particles.

(2) Overlapping α-particles repel one another.

(3) The nuclear forces do not bring about strong correlations whenever two nucleons come very close (say $< 10^{-13}$ cm).

Some comments on assumption (2) are made at the beginning of § 3. Assumption 3 is certainly wrong [10]; however, our discussion will show that the essential features of the α-α interaction are unlikely to be affected by the modifications which are necessary to take the correlations into account.

The discussion given in this paper will show that a number of the difficulties in the way of the α-particle model are much slighter than has often been supposed, and that the verdict about them has to be revised as follows:

Polarisation leads to solution. It is shown in the accompanying paper (ref. [19])) that if the attraction between α-particles were due to Van der Waals forces, then the probability of the α-particles' being dissolved would be small in nuclei in which there are no completely surrounded α-particles.

Additivity. The essential point about additivity is that the additive attractive forces due to V_D and V_{POL} have a larger range than the non-additive part of the repulsive forces due to V_X. This conclusion conflicts to some extent with that reached by Margenau [4]; the reasons for the disagreement are discussed in Appendix 1. The estimates of the expectation values of the non-additive energies which we shall make will show that their magnitudes are not inconsistent with an effectively additive α-α interaction.

The exchange energy calculated in this paper is different from that given by Edwards [6]. The reason for the difference is that we shall separate the exchange energy from the rest of the energy, whereas in ref. [6]) the exchange energy appears as an intimate part of the total energy. The terms which Edwards selects as representing the exchange energy do not coincide with the exchange energy calculated here.

Saturation. The saturation difficulty pointed out by Edwards [6]) arises if one tries to account for the attraction between α-particles without Van der Waals forces. The only source of attraction is then V_D, which operates

as required only if the nucleon-nucleon force has a direct component such as would lead to a violation of the saturation conditions with a monotonic potential. (It is this direct interaction which is responsible for the attraction discovered by Inglis and Edwards.) The saturation difficulty is not an absolute bar to an explanation of the α-α attraction by means of V_D for two reasons: first, if one demands that the nuclear forces should give saturation only within the known range of nuclei, then an attractive V_D is not completely barred; secondly, the essential feature of V_D is that it is due to a nuclear fringing field which extends outside the α-particle so that two α-particles can attract one another without overlapping. The nuclear forces could be modified at short distances so as to give saturation while retaining a tail such as to give the α-particle a fringing field.

The plan of this paper is as follows: in §§ 2, 3 and 4, we shall consider in turn the properties of V_D, V_X and V_{POL}. In § 5 the results are combined and compared with the empirical α-α interactions which have had some success in interpreting the properties of the α-nuclei [11, 12]). The results are discussed in § 6 and the conclusions stated in § 7.

2. The Direct Interaction V_D

2.1. DEFINITION

It can be shown (cf. ref. [8]), eq. (3.27)) that for two α-clusters

$$V_D(\mathbf{X}_{ab}) = \int \mathrm{dv}_{\mathrm{int}}(a)\mathrm{dv}_{\mathrm{int}}(b)\Psi_0^{(\alpha)*}(a)\Psi_0^{(\alpha)*}(b)V_{\alpha\alpha}(a,b)\Psi_0^{(\alpha)}(a)\Psi_0^{(\alpha)}(b) \quad (2.1)$$

where $\Psi_0^{(\alpha)}(a)$ and $\Psi_0^{(\alpha)}(b)$ are the ground state wave functions for the internal motion of α-particles a and b respectively; (we are using the collective labels a, b in the sense $a \equiv (1, 2, 3, 4)$, $b \equiv (5, 6, 7, 8)$, where the symbols 1, 2, . . . 8 denote the co-ordinates of the eight nucleons). $V_{\alpha\alpha}$ is the sum of the nuclear interactions of all the pairs of nucleons which can be constructed by taking one nucleon from each cluster. The symbol $\int \mathrm{dv}_{\mathrm{int}}^{(a)}\mathrm{dv}_{\mathrm{int}}^{(b)}$ signifies integration over the spatial, spin, and isobaric spin co-ordinates of all the nucleons in such a way that the vector $\mathbf{X}_{ab}$ from the mass-centre of the α-cluster a to that of the α-cluster b is held constant. If more than two α-clusters are present, V_D becomes a sum of pair terms of the form (2.1), so that V_D is additive. In writing (2.1) it has been assumed that the fundamental nucleon-nucleon force can be derived from a static potential; the result (2.1) then depends only on the separation $\mathbf{X}_{ab}$ of the two α-clusters and is independent of the relative velocity of the α-clusters.

2.2. NUCLEAR POTENTIAL

To calculate V_D, we assume the interaction potential between nucleons i and j to have the charge-independent form (ref. [13]), Chapters 8, 15)

$$V(ij) = V_C(r_{ij})(a_0+a_\sigma\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+a_\tau\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j+a_{\sigma\tau}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) + V_T(r_{ij})(a_{0T}+a_{\tau T}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)(\boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}_{ij}-\tfrac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j). \tag{2.2}$$

We shall consider two possible forms of spatial dependences in order to get an idea of the sensitivity of the results to the shape of the potential:

Yukawa:

$$V_C(r_{ij}) = V_C\frac{\exp(-r_{ij}/r_C)}{(r_{ij}/r_C)}, \qquad V_T(r_{ij}) = V_T\frac{\exp(-r_{ij}/r_T)}{(r_{ij}/r_T)}; \tag{2.3a}$$

$V_C = 39\,\text{MeV}$, $V_T = 31\,\text{MeV}$, $r_C = 1.186\times10^{-13}\,\text{cm}$, $r_T = 1.532\times10^{-13}\,\text{cm}$,

Gaussian central plus Yukawa tensor:

$$V_C(r_{ij}) = V_C\exp(-\varkappa r_{ij}^2), \qquad V_T(r_{ij}) = V_T\frac{\exp(-r_{ij}/r_T)}{(r_{ij}/r_T)}, \tag{2.3b}$$

$V_C = 44.3\,\text{MeV}$, $V_T = 27.4\,\text{MeV}$, $\varkappa = 0.45\times10^{26}\,\text{cm}^{-2}$, $r_T = 1.63\times10^{-13}\,\text{cm}$,

r_{ij} is the distance between nucleons i and j; $\hat{\mathbf{r}}_{ij}$ is the unit vector pointing from nucleon i to nucleon j. The $\boldsymbol{\sigma}_i$ are $(2/\hbar)$ times the intrinsic mechanical spin operators, and the $\boldsymbol{\tau}_i$ are the isobaric spin operators. The a's are numerical coefficients normalised so that

$$\begin{aligned} a_0+a_\sigma-3a_\tau-3a_{\sigma\tau} &= -1, && \text{(spin triplet state)}\\ a_0-3a_\sigma+a_\tau-3a_{\sigma\tau} &= -q && \text{(spin singlet state)}\\ a_{0T}-3a_{\tau T} &= -3. \end{aligned} \tag{2.4}$$

We take $q = -1.17$ in potential (2.3a) and $q = 1$ in (2.3b). The potential (2.3a) fits the deuteron quadrupole moment, the two-body triplet and singlet scattering lengths, and the triplet effective range. It probably fits the binding energies of the triton and the α-particle [14, 15]). The potential (2.3b) fits the quadrupole moment of the deuteron, the triplet scattering length, and the singlet and triplet effective ranges. For the singlet scattering length, which depends very sensitively on the potential parameters, (2.3b) gives -15.3×10^{-13} cm instead of the experimental [16]) -23.7×10^{-13} cm. Except for the restrictions imposed by (2.4), the two-body data are independent of the exchange character of the potential (2.2). In ref. [15]) the α-particle binding energy was calculated with potentials of the neutral and charge-symmetric exchange types and the two results were found to be practically the same.

From (2.2) we have

$$V_{\alpha x}(a,b) = \sum_{i=1}^{4}\sum_{j=5}^{8} V(ij) = V_{\alpha xC}(a,b)+V_{\alpha xT}(a,b). \tag{2.5}$$

$V_{\alpha xC}$ and $V_{\alpha xT}$ correspond to the central and tensor parts of (2.2) respectively.

2.3. WAVE FUNCTION

To calculate V_D we need $\Psi_0^{(\alpha)}$. If we neglect tensor forces for the moment, then $\Psi_0^{(\alpha)}$ has the form

$$\Psi_{0s}^{(\alpha)}(1234) = \varphi_{0s}^{(\alpha)}(1234)D(1234); \tag{2.6}$$

$\varphi_{0s}^{(\alpha)}(1234)$ is a symmetric function of the spatial co-ordinates, and is invariant with respect to rotations (for which reason we have inserted a suffix s) and translations; $D(1234)$ is a normalised Slater determinant of spin and isobaric spin eigenfunctions:

$$D(1234) = (4!)^{-\frac{1}{2}} \begin{vmatrix} u_+(1)v_+(1) & u_+(2)v_+(2) & u_+(3)v_+(3) & u_+(4)v_+(4) \\ u_+(1)v_-(1) & . & . & . \\ u_-(1)v_+(1) & . & . & . \\ u_-(1)v_-(1) & . & . & u_-(4)v_-(4) \end{vmatrix}. \tag{2.7}$$

The functions $u_\pm$, $v_\pm$ satisfy the equations

$$\begin{aligned} \sigma_{iz}u_\pm(i) &= \pm u_\pm(i), \\ \tau_{iz}v_\pm(i) &= \pm v_\pm(i). \end{aligned} \tag{2.8}$$

As compared with wave functions having different spatial symmetry, (2.6) will give a maximum of potential energy with a minimum of kinetic energy. Additions to the form (2.6) are necessary due to the Coulomb potential, but this effect is small and will be neglected. If we insert (2.6) and (2.7) into (2.1), then with the aid of (2.2) we obtain

$$V_D(\mathbf{X}_{ab}) = 16a_0 \int \mathrm{dv_{int}}(a)\mathrm{dv_{int}}(b)|\varphi_{0s}^{(\alpha)}(1234)|^2\,|\varphi_{0s}^{(\alpha)}(5678)|^2\,V_C(r_{15}); \tag{2.9}$$

the terms in (2.2) which contain a_σ, a_τ, $a_{\sigma\tau}$ do not contribute because

$$\langle D(1234)|(\boldsymbol{\sigma}_i,\ \boldsymbol{\tau}_i,\ \text{or}\ \boldsymbol{\sigma}_i\boldsymbol{\tau}_j)|D(1234)\rangle = 0 \qquad (i, j = 1, 2, 3, 4). \tag{2.10}$$

For monotonic potentials satisfying the saturation conditions, we must have $a_0 \geqq 0$, so that V_D must be repulsive. It follows that V_D can be attractive only if either the potential is monotonic and violates the usual saturation conditions, or if at the least the potential $a_0V_C(r_{ij})$ has an attractive tail; if $a_0V_C(r_{ij})$ has an infinite repulsive core, then the formulae (2.1) and (2.9) become useless because, since $\varphi_{0s}^{(\alpha)}$ is nowhere zero, the integral in (2.9) is infinite. On the other hand a finite repulsive core can be dealt with, although an error might still result from the neglect of spatial correlations due to the repulsion.

For $\varphi_{0s}^{(\alpha)}$ we take the approximate form

$$\varphi_{0s}^{(\alpha)} = \left(\frac{4\beta}{\pi}\right)^{9/4} \exp\left(-\frac{\beta}{2}\sum_{\substack{i,j=1 \\ i>j}}^{4} r_{ij}^2\right); \tag{2.11}$$

this wave function is normalised in the space of the relative spatial co-

ordinates of the four nucleons. The parameter β is related to the root mean square radius ϱ_α of the α-particle by

$$\varrho_\alpha \equiv \overline{\left(\mathbf{r}_1 - \tfrac{1}{4}\sum_{i=1}^{4}\mathbf{r}_i\right)^2}^{\frac{1}{2}} = \frac{3}{4\sqrt{(2\beta)}}. \tag{2.12}$$

2.4. RESULTS

Inserting (2.11) into (2.9), we obtain (dropping the symbols a, b) for the potential (2.3a)

$$V_D(\mathbf{X}) = 16V_C a_0 \cdot \frac{\exp(-X/r_C)}{X/r_C} \cdot \exp(\varrho_\alpha^2/3r_C^2)$$
$$\cdot \left\{ \tfrac{1}{2}\left[1+\operatorname{sgn}\left(\frac{3X}{2\varrho_\alpha^2}-\frac{1}{r_C}\right)\operatorname{erf}\left(\left|-\frac{\varrho_\alpha}{r_C\sqrt{3}}+\frac{X\sqrt{3}}{2\varrho_\alpha}\right|\right)\right] \right. \tag{2.13}$$
$$\left. -\tfrac{1}{2}\exp(2X/r_C)\left[1-\operatorname{erf}\left(\frac{\varrho_\alpha}{r_C\sqrt{3}}+\frac{X\sqrt{3}}{2\varrho_\alpha}\right)\right]\right\},$$

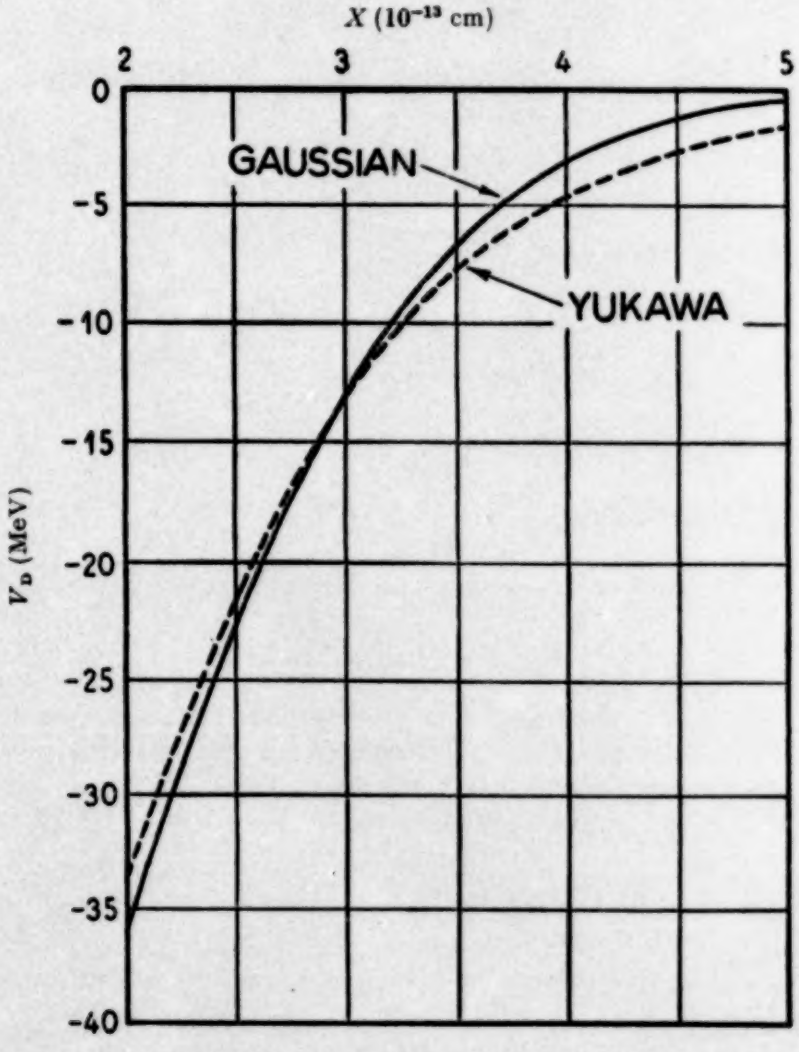

Fig. 2.1. Direct interaction energy V_D.

where sgn (x) is defined to be $+1$ when $x > 0$ and -1 when $x < 0$.

For the r.m.s. radius ϱ_α which appears in (2.13) we have the value $(1.4 \pm 0.15) \times 10^{-13}$ cm obtained from electron scattering at 188 MeV [17]). This radius corresponds to the distance of the centres of the protons from the mass-centre of the α-particle. When we calculate the Coulomb energy of two α-particles (§ 5) we shall need the radius of the charge distribution on the α-particle; this is a little larger at $(1.6 \pm 0.1) \times 10^{-13}$ cm. Most of the calculations were done before the latest radii from electron scattering were available and therefore cover a rather wider range of radii than is now necessary. We evaluate (2.13) for $\varrho_\alpha = 1.1$, 1.4 and 1.8×10^{-13} cm. The complicated factor in curly brackets in (2.13) is slowly varying; for $\varrho_\alpha = 1.5 \times 10^{-13}$ cm, this factor increases from 0.6 at $X = 2 \times 10^{-13}$ cm to 1 for large X so that the value of V_D is given essentially by the factor outside the curly bracket, except at very small separations.

TABLE 2.1

$V_D(X)$-direct interaction.

ϱ_α 10^{-13} cm	X 10^{-13} cm	Yukawa (2.3a) MeV	Gaussian (2.3b) MeV
1.1	2	−33.5	−41.0
	2.5	−18.9	−22.7
	3	−10.7	−11.1
	3.5	− 6.0	− 4.8
	4	− 3.5	− 1.8
	5	− 1.2	− 0.2
1.4	2	−33.8	−36.3
	2.5	−21.5	−22.9
	3	−13.0	−12.9
	3.5	− 7.65	− 6.6
	4	− 4.5	− 3.1
	5	− 1.6	− 0.5
1.8	2	−28.4	−28.3
	2.5	−19.9	−20.0
	3	−13.0	−13.2
	3.5	− 8.3	− 8.0
	4	− 5.0	− 4.5
	5	− 1.7	− 1.1

For the sake of illustration, we choose $a_0 = -0.408$ as for the Serber potential. The results are shown in fig. 2.1 for $\varrho_\alpha = 1.4 \times 10^{-13}$ cm, and in table 2.1 for $\varrho_\alpha = 1.1$, 1.4, 1.8×10^{-13} cm. The sensitivity of the result to the values of the parameters is such that there is a possible error of $\pm 20\%$ if V_C and r_C are to fit the triton and α-particle binding energies [14, 15]) and a possible change of $\pm 16\%$ if the proportion of space exchange force is varied by $\pm 20\%$ of that in the Serber potential.

With the potential (2.3b) we obtain from (2.9) and (2.11)

$$V_D(\mathbf{X}) = 16V_C a_0 \left(1+\frac{3\varkappa}{8\beta}\right)^{-3/2} \exp\left(-\varkappa X^2/1+\frac{3\varkappa}{8\beta}\right). \qquad (2.14)$$

$V_D(\mathbf{X})$ has also been evaluated from formula (2.14), with $a_0 = -0.375$ as for the Serber potential; the results are given in fig. 2.1 for $\varrho_\alpha = 1.4\times 10^{-13}$ cm, and in table 2.1 for $\varrho_\alpha = 1.1,\ 1.4,\ 1.8\times 10^{-13}$ cm.

2.5. THE EFFECT OF TENSOR FORCES

We next discuss the effect of the tensor component of the elementary nucleon-nucleon force (2.2). The first effect of this tensor component is to introduce into the α-particle wave function (2.6) additional terms with non-zero spin and orbital angular momenta coupled so as to give zero total angular momentum. The most important of these terms are D-states [15]). The complete wave function of the α-particle then has the form $(\Psi_{0S}^{(\alpha)}+C\Psi_{0D}^{(\alpha)})/(1+C^2)^{\frac{1}{2}}$, where $\Psi_{0D}^{(\alpha)}$ is a normalised linear superposition of D-states with different spatial symmetries. The total D-state probability $C^2/(1+C^2)$ is of the order of a few percent [15]). It is now easy to see that if one calculates V_D from (2.1) with the modified $\Psi_0^{(\alpha)}$, and includes the tensor part of (2.2) in V_{xx}, then the correction due to the tensor forces which has to be added to the results for central forces shown in fig. 2.1 is of the order of the D-state probability; (all terms containing the D-state probability in lower order than the first vanish because of (2.10)). For a rough estimate of the tensor force correction to V_D, we then have, for the potential (2.3a)

$$\frac{V_D \text{ (due to tensor forces)}}{V_D \text{ (due to central forces)}} \approx (\text{D state probability}) \cdot \frac{V_T}{V_C} \cdot \frac{[\exp(-X/r_T)]/(X/r_T)}{[\exp(-X/r_C)]/(X/r_C)} \cdot \frac{\exp(\varrho_\alpha^2/3r_T^2)}{\exp(\varrho_\alpha^2/3r_C^2)}. \qquad (2.15)$$

(The exponentials containing ϱ_α go some way to taking into account the finite size of the α-particle, as one can see from (2.13).) If we take the D-state probability to be 8 %, and $\varrho_\alpha = 1.4\times 10^{-13}$ cm, then we find from (2.15) that the tensor force correction to V_D is of the order of 8 % of the central force part at $X = 3.7\times 10^{-13}$ cm.

3. The Nucleon Exchange Operator V_X

3.1. INTRODUCTION

V_X has its origin [8, 9]) in the exclusion principle which demands that the wave function be antisymmetric with respect to all pairs of nucleons. Since another expression of the exclusion principle is that no two nucleons with the same spin and charge may be in the same place, and since the α-particle contains a nucleon of each combination of spin and charge, the exclusion

principle demands that α-particles must repel one another when they start to overlap. Therefore V_X must have the general character of a repulsion.

The operation of the exclusion principle in producing this repulsion is illustrated by some results of Skyrme and Perring [18]; they have shown that the $(1s)^4(1p)^4$ shell model wave function for the ground state of Be^8, the $(1s)^4(1p)^8$ for C^{12}, and the $(1s)^4(1p)^{12}$ for O^{16} can be re-arranged into forms which describe the relative motion of 2, 3, and 4 α-particles respectively. They find that the wave functions describing the relative motion of the α-particles vanish at the points where two α-particles coincide; this corresponds to a repulsion between α-particles at short separation.

The repulsion, which is thus a result of a general argument, is consistent with the detailed calculations of the interaction energy of two α-particles made by Margenau [4]) and Edwards [6]), both of whom found positive interaction energies at α-α separations involving appreciable overlap of the α-particles.

In this section we shall estimate the order of magnitude of the contribution of V_X to the expectation value of the energy of a system of α-particles. A difficulty in making such estimates is that the representation for V_X which was derived by Wheeler [9]) and used in ref. [8]) appears to depend explicitly on the total energy. This makes the eigenvalue problem presented by (1.1) rather complicated. To remove this complication, we shall show that V_X can be represented by the solution of an integral equation which does not contain the energy explicitly. We shall then make the integral equation the basis of the discussion of this section.

3.2. THE REPRESENTATION OF V_X

In this subsection we shall develop a method of representing V_X which will be convenient for the subsequent discussion.

We suppose that the wave function of a system of n interacting α-particles has the form

$$\Psi^{(n\alpha)} = \sum_P (\pm) \left[\prod_{i=1}^{n} \Psi_0^{(\alpha)}(a_i) + \Delta\Psi(a_1, \ldots a_n;\ \mathbf{X}_1, \ldots \mathbf{X}_n)\right] F(\mathbf{X}_i, \ldots \mathbf{X}_n) \quad (3.1)$$

where the a_i $(i = 1, 2, \ldots n)$ denote the spin, isobaric spin, and relative spatial co-ordinates of the n sets of four nucleons which make up the n α-particles, and where $\mathbf{X}_i$ is the position vector of the mass-centre of the ith α-particle. $\Psi_0^{(\alpha)}(a_i)$ is the normalised internal ground state wave function of the set a_i of nucleons constituting the ith α-particle; $\Delta\Psi$ $(a_1, \ldots a_n;$ $\mathbf{X}_1, \ldots \mathbf{X}_n)$ represents the distortion of the internal structure of the α-particles due to their mutual polarisation; $\Delta\Psi$ is chosen so that the internal wave function of the α-particles is normalised, i.e. so that

$$\int \left[\prod_{i=1}^{n} \mathrm{dv}_{\mathrm{int}}(a_i)\right] \cdot \left[\prod_{j=1}^{n} \Psi_0^{(\alpha)}(a_j) + \Delta\Psi\right]^* \cdot \left[\prod_{k=1}^{n} \Psi_0^{(\alpha)}(a_k) + \Delta\Psi\right] = 1. \quad (3.2)$$

where the integration $\int \prod_{i=1}^{n} \mathrm{d}v_{\mathrm{int}}(a_i)$ extends only over the internal co-ordinates of the α-particles, while the $\mathbf{X}_i$ are constant. $\Delta\Psi$ is to be constructed so that it vanishes when the separations of the α-particles become large; (see ref. [19])). The function F in (3.1) is the wave function for the relative motion of the α-particles. The summation $\sum_{\mathrm{P}}(\pm)$ signifies that the nucleons are to be distributed among the n α-particles in all possible ways, the resulting wave functions being added with a + or − sign according to whether a particular distribution is obtained by an even or odd number of exchanges from an arbitrarily chosen initial distribution. The internal normalisation condition (3.2) implies that when the α-particles do not overlap (so that the nucleon exchange effects arising from the antisymmetrication $\sum_{\mathrm{P}}$ are small), then the probability density of the α-particle distribution is given by $|F|^2$.

The time-independent wave equation is

$$H\Psi^{(n\alpha)} = E\Psi^{(n\alpha)}, \tag{3.3}$$

where H is the Hamiltonian for the system and E is the energy eigenvalue. In order to formulate V_{X} we have first to obtain a wave equation for the function F. To do this, we divide the $4n$ nucleons up into a particular, arbitrarily chosen, set of n groups of four; we denote these particular groups by $a_1', a_2', \ldots a_n'$, and their mass-centres by $\mathbf{X}_1', \mathbf{X}_2', \ldots \mathbf{X}_n'$. We then insert $\Psi^{(n\alpha)}$ from (3.1) into (3.3), multiply on the left by $[\prod_{i=1}^{n} \Psi_0^{(\alpha)}(a_i') + \Delta\Psi(a_1', \ldots a_n'; \mathbf{X}_1', \ldots \mathbf{X}_n')]^*$ and integrate over all co-ordinates (including spins and isobaric spins) except the vectors $\mathbf{X}_1', \mathbf{X}_2', \ldots \mathbf{X}_n'$. We denote these integrations by $\int \mathrm{d}\mathbf{u}$. In those terms which have the same distribution of nucleons over α-particles in both wavefunction factors, $\int \mathrm{d}\mathbf{u}$ denotes what we have written $\int \prod_{i=1}^{n} \mathrm{d}v_{\mathrm{int}}(a_i)$ in (3.2); in the terms which contain different distributions a_i' and a_i on the left and right, $\int \mathrm{d}\mathbf{u}$ denotes integration over everything except $\mathbf{X}_1', \ldots \mathbf{X}_n'$ and $\mathbf{X}_1, \ldots \mathbf{X}_n$, the final integration over the $\mathbf{X}_i$ being then written out explicitly. (Certain precautions have to be taken in this procedure. These are described in Appendix 2.) In this way, equation (3.3) becomes

$$\begin{aligned}
&\int \mathrm{d}\mathbf{u} \Big[\prod_{i=1}^{n} \Psi_0^{(\alpha)}(a_i')\Big]^* H \Big[\prod_{j=1}^{n} \Psi_0^{(\alpha)}(a_j')\Big] F(\mathbf{X}_1', \ldots \mathbf{X}_n') \qquad H_{\mathrm{D}}\ F \\
&+ \Big[\int \mathrm{d}\mathbf{u} \Delta\Psi(a_1', \ldots a_n'; \mathbf{X}_1', \ldots \mathbf{X}_n')^* \\
&\qquad\qquad H \prod_{i=1}^{n} \Psi_0^{(\alpha)}(a_i') + \text{c.c.}\Big] F(\mathbf{X}_1, \ldots \mathbf{X}_n') \\
&+ \Big[\int \mathrm{d}\mathbf{u} \Delta\Psi(a_1', \ldots a_n'; \mathbf{X}_1', \ldots \mathbf{X}_n')^* \\
&\qquad H\Delta\Psi(a_1', \ldots a_n'; \mathbf{X}_1', \ldots \mathbf{X}_n')\Big] F(\mathbf{X}_1', \ldots \mathbf{X}_n')
\end{aligned} \Bigg\} + V_{\mathrm{POL}} F \tag{3.4}$$

$$\left.\begin{aligned} &+\sum_{\text{exchanges}}(\pm)\int\Big(\prod_{i=1}^{n}\mathrm{d}\mathbf{X}_i\Big)\mathrm{d}\mathbf{u}\Big[\prod_{j=1}^{n}\Psi_0^{(\alpha)}(a_j')+\Delta\Psi(a_1',\ldots a_n';\ \mathbf{X}_1',\ldots,\mathbf{X}_n')\Big]^* \\ &\cdot(H-E)\cdot\Big[\prod_{k=1}^{n}\Psi_0^{(\alpha)}(a_k)+\Delta\Psi(a_1,\ldots a_n;\ \mathbf{X}_1,\ldots\mathbf{X}_n)\Big]F(\mathbf{X}_1,\ldots\mathbf{X}_n) \end{aligned}\right\}+V_{\mathrm{X}}F$$

$$= EF(\mathbf{X}_1',\ldots\mathbf{X}_n'). \qquad = EF$$

The symbols on the right of (3.4) are abbreviations for the terms standing opposite to them on the left. In the terms $V_{\mathrm{X}}F$, we have collected all terms where the grouping of particles into sets of four in the right hand factor is different from the grouping a_1', a_2', $\ldots a_n'$.

It is convenient to make some further decompositions and transformations in the terms of (3.4). First of all, in $H_{\mathrm{D}}F$ we decompose H as follows:

$$H=\sum_{i=1}^{n}K_i'+\sum_{i=1}^{n}H_{\mathrm{int}\,i}'+\sum_{\substack{i,\,j=1\\ i>j}}^{n}V_{\alpha\alpha}(a_i',\ a_j'). \tag{3.5}$$

where K_i' is the kinetic energy of the mass-centre of the four nucleons a_i', $H_{\mathrm{int}\,i}'$ is the internal energy of the α-particle built up from them, and $V_{\alpha\alpha}$ $(a_i',\ a_j')$ is the interaction potential between the set a_i' on the one hand and the set a_j' on the other. Using (3.5), we obtain

$$(H_{\mathrm{D}}F)(\mathbf{X}_1',\ldots\mathbf{X}_n')=\Big[\sum_{i=1}^{n}K_i'+\sum_{\substack{i,\,j=1\\ i>j}}^{n}V_{\mathrm{D}}(\mathbf{X}_{a_i'\,a_j'})+nE_0^{(1)}\Big]F(\mathbf{X}_1',\ldots\mathbf{X}_n'), \tag{3.6}$$

where $E_0^{(1)}$ is the internal ground state energy of a single α-particle, and where $V_{\mathrm{D}}(\mathbf{X}_{a_i'\,a_j'})$ is defined as in equation (2.1). If we insert (3.6) into (3.4), we get equation (1.1) which was derived in ref. [8]) from a variational principle. The derivation from the variational principle has the advantage that it tells one how to choose the form of $\Delta\Psi$; it also makes clear that the eigenvalues E which one obtains by solving (3.4) for F are approximations to the exact eigenvalues of (3.3) in the sense of Rayleigh's principle with the wave function (3.1).

We next transform $V_{\mathrm{X}}F$ so as to remove the explicit dependence on E. In doing this, we shall make the approximation of neglecting all polarisation effects. This approximation implies that we shall neglect the terms $\Delta\Psi$ in the two factors for the internal wave function. We write H in $V_{\mathrm{X}}F$ in the form (3.5), with the difference that in H the nucleons are now to be divided into sets of four as in the internal wave function factor on the right of $(H-E)$. We obtain

$$\begin{aligned}(V_{\mathrm{X}}F)(\mathbf{X}_1',\ldots\mathbf{X}_n')=&\sum_{\text{exchanges}}(\pm)\int\Big(\prod_{i=1}^{n}\mathrm{d}\mathbf{X}_i\Big)\mathrm{d}\mathbf{u}\Big[\prod_{j=1}^{n}\Psi_0^{(\alpha)}(a_j')\Big]^* \\ &\cdot\Big[\sum_{k=1}^{n}K_k+\sum_{k=1}^{n}H_{\mathrm{int}\,k}+\sum_{\substack{k,\,l=1\\ k>l}}^{n}V_{\alpha\alpha}(a_k,a_l)-E\Big]\cdot\Big[\prod_{m=1}^{n}\Psi_0^{(\alpha)}(a_m)\Big]F(\mathbf{X}_1,\ldots\mathbf{X}_n).\end{aligned} \tag{3.7}$$

We now substitute for the factor $[\sum K + \ldots]F$ from (1.1), drop the polarisation term $V_{\mathrm{POL}}F$, and thus obtain an integral equation for $(V_{\mathbf{X}}F)$. This integral equation does not contain the energy explicitly.

We now define an integral kernel corresponding to $V_{\mathbf{X}}$ by the equation

$$(V_{\mathbf{X}}F)(\mathbf{X}_1', \ldots \mathbf{X}_n') \equiv \int d\mathbf{X}_1 \ldots d\mathbf{X}_n V_{\mathbf{X}}(\mathbf{X}_1', \ldots \mathbf{X}_n'; \mathbf{X}_1, \ldots \mathbf{X}_n) F(\mathbf{X}_1, \ldots \mathbf{X}_n) \tag{3.8}$$

If we use this representation in the integral equation for $(V_{\mathbf{X}}F)$, then we can obtain an integral equation for $V_{\mathbf{X}}(\mathbf{X}_1', \ldots \mathbf{X}_n'; \mathbf{X}_1, \ldots \mathbf{X}_n)$ alone by dropping the factors F, on the assumption that the eigenfunctions of (1.1) form a complete set. The integral equation for $V_{\mathbf{X}}(\mathbf{X}_1', \ldots \mathbf{X}_n', \mathbf{X}_1, \ldots \mathbf{X}_n)$ is

$$V_{\mathbf{X}}(\mathbf{X}_1', \ldots \mathbf{X}_n'; \mathbf{X}_1 \ldots \mathbf{X}_n) + \int \left(\prod_{i=1}^{n} d\mathbf{X}_i''\right) V_{\mathbf{X}}(\mathbf{X}_1'', \ldots \mathbf{X}_n''; \mathbf{X}_1 \ldots \mathbf{X}_n)$$
$$\left\{ \sum_{\text{exchanges}} (\pm) \int d\mathbf{u} \left[\prod_{j=1}^{n} \Psi_0^{(\alpha)}(a_j')\right]^* \left[\prod_{k=1}^{n} \Psi_0^{(\alpha)}(a_k'')\right] \right\} \tag{3.9}$$
$$= \sum_{\text{exchanges}} (\pm) \int d\mathbf{u} \left[\prod_{i=1}^{n} \Psi_0^{(\alpha)}(a_i')\right]^* \left\{ \sum_{j>k} [V_{\alpha\alpha}(a_j, a_k) - V_{\mathrm{D}}(\mathbf{X}_{a_j a_k})] \right\} \left[\prod_{l=1}^{n} \Psi_0^{(\alpha)}(a_l)\right].$$

Let us denote the right hand side by $P(\mathbf{X}', \mathbf{X})$, and the contents of the { } bracket on the left by $Q(\mathbf{X}', \mathbf{X}'')$. Equation (3.9) then becomes formally

$$(1+Q)V_{\mathbf{X}} = P. \tag{3.10}$$

Equations (3.9) and (3.10) form the basis of the following discussion.

3.3. APPROXIMATION OF $V_{\mathbf{X}}$ FOR SMALL OVERLAP

One of the conditions of self-consistency of the α-particle model is that the α-particles must not overlap too much. It must therefore be a good approximation to take into account only terms with the smallest number of exchange integrals in the expectation value of the nucleon exchange energy. Loosely speaking, one may say that this approximation takes into account only the exchange of single pairs of nucleons. This means that we can neglect the term $QV_{\mathbf{X}}$ on the left of (3.10) because it contains more exchange integrals than does $V_{\mathbf{X}}$ itself. In P, we shall retain only those terms which arise from single pair exchanges. We shall denote the resulting approximation for $V_{\mathbf{X}}$ by $V_{\mathbf{X}}^{(1)}$.

Some support for the single-pair-exchange approximation comes from Edwards [6]) and Margenau's [4]) results for two α-particles. Edwards finds that for all values of the mean separation, the magnitude of the double exchange integrals is between one third and one tenth of that of the single exchange integrals (see table 1 of reference [6])). It is not difficult to verify that the single-pair exchange approximation to Margenau's results (given in his fig. 2) is almost perfect when the separation of the centres of his

two α-particles is larger than the radius of an α-particle. (Radius here means the radius of a sphere of uniform density which has the same root mean square radius as the α-particle.)

It has to be emphasised that $V_X^{(1)}$ will be used in place of V_X only for the estimate of the expectation value of the energy, in other words for the estimate of the magnitude of V_X where the wave function for the relative motion of α-particles is large and overlap is small. It is of course impossible to replace V_X by $V_X^{(1)}$ when the overlap is large; we shall assume that this is not important for the expectation value because the wave function is small when the overlap is large.

3.4. DIVISION OF V_X INTO ADDITIVE AND NON-ADDITIVE PARTS

We shall study the additivity properties in the $V_X^{(1)}$ approximation. We start by writing down $V_X^{(1)}$ from (3.9) and re-arranging the summation so as to make evident the contributions from the different α-particles. Thus we have

$$V_X^{(1)}(\mathbf{X}_1', \ldots \mathbf{X}_n'; \mathbf{X}_1, \ldots \mathbf{X}_n) =$$
$$- \sum_{\substack{\text{single}\\ \text{exchanges}}} \int \mathrm{d}\mathbf{u} \left[\prod_{i=1}^{n} \Psi_0^{(\alpha)}(a_i')\right]^* \left\{\sum_{j>k} [V_{\alpha\alpha}(a_j, a_k) - V_D(\mathbf{X}_{a_j a_k})]\right\} \left[\prod_{l=1}^{n} \Psi_0^{(\alpha)}(a_l)\right] \tag{3.11}$$
$$= -16 \sum_{\substack{i>j\\ \text{(pairs of α's)}}} \int \mathrm{d}\mathbf{u}(a_i, a_j; a_i', a_j') \Psi_0^{(\alpha)*}(a_i') \Psi_0^{(\alpha)*}(a_j') \{[V_{\alpha\alpha}(a_i, a_j) - V_D(\mathbf{X}_{a_i a_j})]$$
$$+ \sum_{k \neq i,j} [\overline{V_{\alpha\alpha}(a_k, a_i)}|_k - V_D(\mathbf{X}_{a_k a_i}) + \overline{V_{\alpha\alpha}(a_k a_j)}|_k - V_D(\mathbf{X}_{a_k a_j})]\} \Psi_0^{(\alpha)}(a_i) \Psi_0^{(\alpha)}(a_j).$$

In (3.11) the symbols (a_i, a_j, a_i', a_j') signify the grouping of a particular set of eight nucleons into two α-particles in two ways which differ by the exchange of a particular single pair of nucleons (the number of such exchanges possible is allowed for by the factor (16)). The terms in the interaction potentials $V(a_i, a_j)$ etc. are to be selected in accordance with the grouping of nucleons in the factor $\Psi_0^{(\alpha)}(a_i)\Psi_0^{(\alpha)}(a_j)$ at the right of the integral. The notation $\overline{V_{\alpha\alpha}(a_k, a_i)}|_k$ means

$$\overline{V_{\alpha\alpha}(a_k, a_i)}|_k \equiv \int \mathrm{d}v_{\mathrm{int}}(a_k) \Psi_0^{(\alpha)*}(a_k) V_{\alpha\alpha}(a_k, a_i) \Psi_0^{(\alpha)}(a_k), \tag{3.12}$$

the integration being over the internal co-ordinates of α-particle a_k. The integration $\int \mathrm{d}\mathbf{u}(a_i, a_j; a_i', a_j')$ signifies integration over all the co-ordinates of the eight nucleons constituting the α-particles a_i', a_j' (and their new versions a_i, a_j), in such away that $(\mathbf{X}_i', \mathbf{X}_j', \mathbf{X}_i, \mathbf{X}_j)$ are all held constant.

The individual terms under the Σ sign in (3.11) can be arranged in a different way which makes their meaning clear. The term for a particular set (i, j, k) is the sum over the eight nucleons comprising the two α-particles which exchange a nucleon of the difference between the potential energy

in the field of the odd α-particle, and the average of this potential energy for a nucleon distributed half over one and half over the other α-particle.

In (3.11) all potential terms except those written down disappear on integration. The remaining terms are of the following two types. The contents of the first [ ] bracket in the integrand concern only the two α-particles which take part in a nucleon exchange; this term therefore represents an additive interaction. The terms in the sum Σ [ ] each concern the two α-particles which exchange a nucleon, and one other α-particle; these terms therefore represent a non-additive interaction. For a nuclear potential of the type (2.2) and an S state α-particle wave function of the type (2.6), only the a_0 part of the potential contributes to the non-additive terms (because of (2.10) and (3.12)). Thus if $a_0 = 0$ (as in the Rosenfeld mixture), then $V_X^{(1)}$ is additive.

The origin of the possibility that the major part of the nucleon exchange energy may be additive lies in the antisymmetrisation through the summation $\sum_P(\pm)$ in (3.1), which could be re-arranged into a summation over pairs of α-particles in (3.11). The wave function can be antisymmetrised in this way only because the α-particle sub-units are closed shells. If the sub-units are not closed shells, as in atoms with valence electrons, this method of antisymmetrisation does not give the correct wave function and the (electron) exchange interaction is not additive.

The conclusion that the nucleon exchange energy can under certain circumstances be additive has also been reached by Nesbet [20]) from another point of view. Nesbet considered the ground state of nuclei. He constructs the wave function from orbitals which are spatially localised and finds that the energy of interaction of groups of orbitals localised in different places can be expressed as a sum of terms each of which involves only a pair of orbitals. The result is based on the assumption that the occupied orbitals are spatially localised, and that the exclusion principle acts through the requirement that all orbitals be orthogonal. Nesbet does not construct such a set of orbitals. (One way in which they can be constructed is to confine the different orbitals to the interiors of a set of spatial cells which do not overlap. There is then no nucleon exchange energy at all.)

The physical origin of the non-additive terms is suggested by the fact that only the a_0 part of the nuclear potential contributes to the non-additive effect and is probably as follows: If $a_0 \neq 0$, the α-particle is surrounded by a fringing field which extends outside the mass-distribution. The fringing field affects the height of the potential barriers between nearby α-particles, and therefore the rate at which nucleons are exchanged by the tunnelling effect. The origin of the non-additive terms is thus quite different from the effect by which electron exchange forces can bind two hydrogen atoms together into a molecule, but not three.

3.5. THE MAGNITUDE OF V_X FOR TWO α-PARTICLES

We shall now estimate the magnitude of V_X. Such an estimate is difficult because V_X is not a static potential. However, all we wish to do is to establish that there is a range of α-α separations where the attractive interactions (V_D+V_{POL}) predominate over the repulsive V_X; for this purpose a rough estimate of V_X will be sufficient, and we shall obtain such an estimate by calculating the expectation value of $V_X^{(1)}$ with a wave function of Margenau's form [4]). We shall first discuss the case of two α-particles; in § 3.6 we shall estimate the magnitude of the non-additive component in $V_X^{(1)}$.

We consider the wavefunction

$$F^{(2\alpha)}(\mathbf{X}_1, \mathbf{X}_2) = \left(\frac{16\beta}{\pi}\right)^{3/8} \cdot \frac{1}{\sqrt{2}} \cdot [1+\exp(-8\beta(\mathbf{A}-\mathbf{B})^2)]^{-\frac{1}{4}} \tag{3.13a}$$

$$\times \{\exp[-8\beta(\mathbf{X}_1-\mathbf{A})^2-8\beta(\mathbf{X}_2-\mathbf{B})^2] + \exp[-8\beta(\mathbf{X}_1-\mathbf{B})^2-8\beta(\mathbf{X}_1-\mathbf{A})^2]\}.$$

A and **B** are two fixed points. The reason for the choice of the coefficient 8β will appear later. $F^{(2\alpha)}$ is normalised so that $\int d\mathbf{X}_1\, d\mathbf{X}_2 |F^{(2\alpha)}(\mathbf{X}_1, \mathbf{X}_2)|^2 = 1$. Because of its symmetry, the wave-function (3.13a) cannot even approximate to an eigenfunction of the Hamiltonian for two α-particles, and therefore describes a transient configuration. (It would have a real physical meaning only if it were taken to represent two adjacent α-particles in a large lattice.)

We now calculate the expectation value

$$\langle F^{(2\alpha)} | V_X^{(1)} | F^{(2\alpha)} \rangle$$

$$\equiv \int d\mathbf{X}_1'\, d\mathbf{X}_2'\, d\mathbf{X}_1\, d\mathbf{X}_2\, F^{(2\alpha)*}(\mathbf{X}_1', \mathbf{X}_2') V_X^{(1)}(\mathbf{X}_1', \mathbf{X}_2'; \mathbf{X}_1, \mathbf{X}_2) F^{(2\alpha)}(\mathbf{X}_1, \mathbf{X}_2).$$

From eq. (3.11), and remembering that we are dealing with only two α-particles, we get

$$\langle F^{(2\alpha)} | V_X^{(1)} | F^{(2\alpha)} \rangle$$

$$= -16 \int d\mathbf{u}\, d\mathbf{X}_1'\, d\mathbf{X}_2'\, d\mathbf{X}_1\, d\mathbf{X}_2\, F^{(2\alpha)*}(\mathbf{X}_1', \mathbf{X}_2') \Psi_0^{(\alpha)*}(a_1') \Psi_0^{(\alpha)*}(a_2')$$

$$\times [V_{\alpha\alpha}(a_1, a_2) - V_D(\mathbf{X}_{a_1 a_2})] F^{(2\alpha)}(\mathbf{X}_1, \mathbf{X}_2) \Psi_0^{(\alpha)}(a_1) \Psi_0^{(\alpha)}(a_2). \tag{3.14}$$

We shall choose $a_1 \equiv (1234)$, $a_2 \equiv (5678)$, $a_1' \equiv (1235)$, $a_2' \equiv (4678)$. If we neglect polarisation and use the approximation (2.6), (2.7), (2.11) for $\Psi_0^{(\alpha)}$, we can write

$$F^{(2\alpha)}(\mathbf{X}_1, \mathbf{X}_2) \Psi_0^{(\alpha)}(a_1) \Psi_0^{(\alpha)}(a_2) = 8 \cdot \left(\frac{4\beta}{\pi}\right)^6 \frac{1}{\sqrt{2}} [1+\exp(-8\beta(\mathbf{A}-\mathbf{B})^2)]^{-\frac{1}{4}}$$

$$\times \left\{ \exp\left[-2\beta \sum_{i=1}^{4} (\mathbf{r}_i-\mathbf{A})^2 - 2\beta \sum_{j=5}^{8} (\mathbf{r}_j-\mathbf{B})^2\right] \right. \tag{3.13b}$$

$$\left. + \exp\left[-2\beta \sum_{i=1}^{4} (\mathbf{r}_i-\mathbf{B})^2 - 2\beta \sum_{j=5}^{8} (\mathbf{r}_j-\mathbf{A})^2\right]\right\}$$

$$\times D(1234) D(5678).$$

$F^{(2\alpha)}(\mathbf{X}_1', \mathbf{X}_2')\Psi_0^{(\alpha)}(a_1')\Psi_0^{(\alpha)}(a_2')$ is a similar expression with the coordinates of nucleons 4 and 5 interchanged. To evaluate the integral (3.14), we write the wavefunctions appearing in the integrand in the form (3.13b), and replace the volume element $\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{X}_1' \ldots \mathrm{d}\mathbf{X}_2$ by $\mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_8 \times 2^{-6}$. The factor 2^{-6} comes from the Jacobian. We shall neglect the crossterms arising from the two exponentials in the { } bracket in (3.13b); these terms contain triple exchange integrals, while the terms we are retaining contain only single exchanges. We shall also drop the exponential in the normalisation factor $[1+\exp \ldots]^{-\frac{1}{2}}$; if we put $8\beta = (2\varrho_\alpha/3)^{-2}$ and $\varrho_\alpha = 1.5\times10^{-13}$ cm, then the value of this exponential is 0.02 for $|\mathbf{A}-\mathbf{B}| = 2\times10^{-13}$ cm.

Because of the complicated form of even $V_{\mathbf{X}}^{(1)}$, we shall use a pure Gaussian potential for the fundamental interaction between nucleons; i.e. in place of the potential (2.2), we assume

$$V(ij) = V_C e^{-\varkappa r_{ij}^2}(a_0+a_\sigma \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+a_\tau \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j+a_{\sigma\tau}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j). \quad (3.15)$$

The parameters a_0, a_σ, a_τ, $a_{\sigma\tau}$ are to be normalised as in (2.4). We shall choose $V_C = 72$ MeV, $\varkappa = 0.45\times10^{26}$ cm^{-2}, and $q = 0.62$. These parameters fit the singlet and triplet scattering lengths and the triplet effective range in the two-nucleon problem [21, 22].

Substituting (3.13b) into (3.14) and using the definition (2.1) and (2.5), we get with the aid of (3.14) and (3.15)

$$\begin{aligned}
&\langle F^{(2\alpha)}|V_{\mathbf{X}}^{(1)}|F^{(2\alpha)}\rangle \\
&= -\frac{16V_C}{\left(1+\frac{\varkappa}{2\beta}\right)^{3/2}}\left\{(a_\sigma+a_\tau+3a_{\sigma\tau})\left[\frac{3}{4}\exp\left(-2\beta\frac{1+\frac{\varkappa}{\beta}}{1+\frac{\varkappa}{2\beta}}X^2\right)\right.\right. \\
&\qquad\left. -\frac{3}{2}\exp\left(-2\beta\frac{1+\frac{5}{8}\frac{\varkappa}{\beta}}{1+\frac{\varkappa}{2\beta}}X^2\right)+\frac{3}{4}\exp(-2\beta X^2)\right] \\
&+a_0\left[\frac{9}{4}\exp\left(-2\beta\frac{1+\frac{\varkappa}{\beta}}{1+\frac{\varkappa}{2\beta}}X^2\right)+\frac{3}{2}\exp\left(-2\beta\frac{\left(1+\frac{5}{8}\frac{\varkappa}{\beta}\right)}{\left(1+\frac{\varkappa}{2\beta}\right)}\right)\right. \\
&\qquad\left.\left. +\frac{1}{4}\exp(-2\beta X^2)-4\exp\left(-2\beta\frac{\left(1+\frac{25}{32}\frac{\varkappa}{\beta}\right)}{\left(1+\frac{\varkappa}{2\beta}\right)}X^2\right)\right]\right\},
\end{aligned} \quad (3.16)$$

where $\mathbf{X} = \mathbf{A}-\mathbf{B}$.

The way in which the size of the α-particle determines the range of $V_{\mathbf{X}}$ shows up in the domination of the ranges of the terms in (3.16) by the constant β.

It can be seen from (3.16) that the value of $\langle V_X^{(1)} \rangle$ is kept down by heavy cancellation between the different terms. In particular, $\langle V_X^{(1)} \rangle$ vanishes if $\varkappa \ll \beta$. Although the two-pair exchange terms on the right of (3.9) are not taken into account in $V_X^{(1)}$ it has been checked that there is very heavy cancellation in these terms too; they too cancel completely when $\varkappa \ll \beta$.

We see from (3.16) that the parameters a occur only in the combination a_0 and $(a_\sigma + a_\tau + 3a_{\sigma\tau})$. If we have $a_0 = 0$ (e.g. the Rosenfeld mixture), then it follows from (2.4) that $(a_\sigma + a_\tau + 3a_{\sigma\tau}) = \frac{1}{2}(1+q) = +0.81$. In the Serber potential we have $a_0 = -0.303$, and $(a_\sigma + a_\tau + 3a_{\sigma\tau}) = +0.506$.

The magnitude of $\langle V_X^{(1)} \rangle$ is given in table 3.1 for a potential (3.15) with $a_0 = 0$, and in table 3.2 for the Serber potential.

TABLE 3.1

$\langle V_X^{(1)} \rangle$ for two α-particles; central Gaussian potential with $a_0 = 0$ (in MeV)

$X(10^{-13}$ cm)	$\varrho_\alpha = 1.4 \times 10^{-13}$ cm	1.55	1.7
2.5	5.30	6.21	6.56
3.0	1.83	2.82	3.61
3.5	0.26	0.77	1.40
4.0	−0.12	−0.02	+0.25
4.5	−0.11	−0.16	−0.13

TABLE 3.2

$\langle V_X^{(1)} \rangle$ for two α-particles; central Gaussian Serber potential (in MeV).

$X(10^{-13}$ cm)	$\varrho_\alpha = 1.4 \times 10^{-13}$ cm	1.55	1.7
2.5	3.05	3.24	3.14
3.0	1.64	2.09	2.31
3.5	0.65	1.05	1.39
4.0	0.19	0.41	0.67
4.5	0.03	0.11	0.25

A rough estimate of the order of magnitude of the multiple pair exchange energy can be made by noting that according to Edwards [6]) the multiple pair exchange terms in the energy are smaller than the single-pair-exchange terms by a factor of 0.1 to 0.3. By computing the interaction energy to lowest order in the overlap integrals from Margenau's paper [4]) and comparing the results with Margenau's $(\Delta_1 E)$, which is closely related to our V_X, one finds that the multiple pair terms give contributions smaller by more than a factor five than the single-pair exchanges if the overlap is not too large. If we take $X = 3.75 \times 10^{-13}$ cm, corresponding to the moment of inertia of C^{12} as determined by the first (4.44 MeV, 2^+) excited state [11]), and $\varrho_\alpha = 1.4 \times 10^{-13}$ cm, and divide the corresponding figures in tables 3.1 and 3.2 by five, then we get a multiple-pair-exchange energy of the order

of 0.1 MeV. This is small in comparison with the α-α binding energy of about 2 MeV per nearest neighbour bond in the α-nuclei, and rather less than the deviations of the binding energy per bond from a constant; (see ref. [1]), table 2.1, col. 10).

Before leaving the nucleon exchange energy for two α-particles, we have to remark that when $a_0 \neq 0$, there must be a component of V_X which cancels the enormous magnitude reached by V_D at very short separation. (For example, for $\varrho_\alpha = 1.55\times 10^{-13}$ cm and the potential (2.3b), equation (2.14) gives 53 MeV at $X = 0$ with the Serber potential.) The reason for the occurrence of these large values is that the exclusion principle is only taken into account if one considers V_D and V_X together. At short α-α separations V_D gives the direct interaction between two superposed α-particles; however, the exclusion principle prohibits this superposition, so that when two α-clusters come close together, then one must dissolve; the eight nucleons then occupy a much larger volume than that of any single α-particle and the range factor in the a_0 part of the nuclear potential operates to reduse the a_0 potential energy from its value for complete superposition.

3.6. NON-ADDITIVE TERMS IN $V_X^{(1)}$.

We next estimate the order of magnitude of the non-additive terms in $V_X^{(1)}$ (eq. 3.11)). The non-additive terms each concern three α-particles. Since the forces have a short range, it will be a good approximation to consider only triplets of neighbouring α's so disposed that the members of a triplet form roughly an equilateral triangle. We shall calculate the non-additive energy associated with a single triplet.

We consider a wavefunction of the form (3.13a) generalised to three α-particles vibrating about three fixed points **A**, **B**, and **C** at the apices of an equilateral triangle; i.e. we take

$$F^{(3\alpha)}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = N\left(\frac{16\beta}{\pi}\right)^{3/4}\cdot\frac{1}{\sqrt{6}}\sum_P \exp\{-8\beta[(\mathbf{X}_1-\mathbf{A})^2+(\mathbf{X}_2-\mathbf{B})^2+(\mathbf{X}_3-\mathbf{C})^2]\}. \quad (3.17a)$$

The sum $\sum_P$ contains all six terms corresponding to different assignments of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ to the points (**A**, **B**, **C**). N is a normalisation constant analogous to the factor $[1+\exp\ldots]^{-\frac{1}{2}}$ in (3.13a), and ensures that $\int d\mathbf{X}_1\, d\mathbf{X}_2\, d\mathbf{X}_3\, |F(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)|^2 = 1$. As in § 3.5, we shall put $N = 1$, i.e. we shall neglect cross terms arising from the different terms in $\sum_P$.

We calculate the expectation value of the non-additive part of $\langle F^{(3\alpha)}|V_X^{(1)}|F^{(3\alpha)}\rangle$, and denote this by appending a suffix na. From eq. (3.11), and taking into account the symmetry of $F^{(3\alpha)}$ between $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, we get

$$\langle F^{(3\alpha)}|V_X^{(1)}|F^{(3\alpha)}\rangle_{\text{na}} =$$

$$= -3.16\int \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{X}_1'\,\mathrm{d}\mathbf{X}_2'\,\mathrm{d}\mathbf{X}_3'\,\mathrm{d}\mathbf{X}_1\,\mathrm{d}\mathbf{X}_2\,\mathrm{d}\mathbf{X}_3\, F^{(3\alpha)*}(\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3') \prod_{i=1}^{3} \Psi_0^{(\alpha)*}(a_i')$$
$$\{[V_{\alpha\alpha}(a_1, a_3) - V_D(\mathbf{X}_{a_1 a_3})] + [V_{\alpha\alpha}(a_2, a_3) - V_D(\mathbf{X}_{a_2 a_3})]\} \qquad (3.18)$$
$$F^{(3\alpha)}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \prod_{i=1}^{3} \Psi_0^{(\alpha)}(a_i).$$

We shall choose

$$a_1 \equiv (1234), \quad a_2 \equiv (5678), \quad a_3 \equiv (9101112),$$
$$a_1' \equiv (1235), \quad a_2' \equiv (4678), \quad a_3' \equiv a_3.$$

We have

$$\mathbf{X}_{a_1 a_3} = \mathbf{X}_1 - \mathbf{X}_3, \qquad \mathbf{X}_{a_2 a_3} = \mathbf{X}_2 - \mathbf{X}_3.$$

If we neglect polarisation and use the approximate form (2.6), (2.7), and (2.11) for $\Psi_0^{(\alpha)}$, then we have

$$F^{(3\alpha)}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \prod_{i=1}^{3} \Psi_0^{(\alpha)}(a_i) =$$
$$N \cdot \left(\frac{4\beta}{\pi}\right)^9 \cdot \frac{2^{9/2}}{\sqrt{6}} \sum_{\mathrm{P}} \exp\left\{-2\beta\left[\sum_{i=1}^{4}(\mathbf{r}_i - \mathbf{A})^2 + \sum_{j=5}^{8}(\mathbf{r}_j - \mathbf{B})^2 + \sum_{k=9}^{12}(\mathbf{r}_k - \mathbf{C})^2\right]\right\} \qquad (3.17\text{b})$$
$$\times D(1234)\, D(5678)\, D(9101112).$$

The expression for $F^{(3\alpha)}(\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3') \prod_{i=1}^{3} \Psi_0^{(\alpha)}(a_i')$ is similar, except that all the co-ordinates of nucleons 4 and 5 are exchanged. To evaluate the integral in (3.18), we express the integrand as a function of $\mathbf{r}_1 \ldots \mathbf{r}_{12}$, and replace the volume element $\mathbf{u}\,\mathrm{d}\mathbf{X}_1' \ldots \mathrm{d}\mathbf{X}_3$ by $\mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_{12} \times 2^{-9}$, where the factor 2^{-9} comes from the Jacobian. In the calculation, we shall neglect cross terms arising from the different terms in $\sum_{\mathrm{P}}$ in (3.17b) because they contain more overlap integrals than do the direct terms.

For the nuclear potential we take (2.3b). Because of (2.10), only the a_0 part contributes. The result is

$$\langle F^{(3\alpha)}|V_X^{(1)}|F^{(3\alpha)}\rangle_{\text{na}} = a_0 V_C \left(1 + \frac{\varkappa}{2\beta}\right)^{-3/2} \cdot 96$$
$$\times \left\{-3 \exp\left[-2\beta \frac{1 + \frac{\varkappa}{\beta}}{1 + \frac{\varkappa}{2\beta}} X^2\right] - \exp\left[-2\beta \frac{1 + \frac{7\varkappa}{8\beta}}{1 + \frac{\varkappa}{2\beta}} X^2\right]\right. \qquad (3.19)$$
$$\left. + 4 \exp\left[-2\beta \frac{1 + \frac{121}{128}\frac{\varkappa}{\beta}}{1 + \frac{\varkappa}{2\beta}} X^2\right]\right\}.$$

The values of (3.19) are given in table 3.3 for the Serber exchange type. X is the length of the side of the equilateral triangle **A**, **B**, **C**.

TABLE 3.3

$\langle V_X^{(1)} \rangle_{\text{non-additive}}$ for three α-particles at the apices of an equilateral triangle of length of side X.

$X(10^{-13}$ cm)	$\varrho_\alpha = 1.4\times10^{-13}$ cm	1.55	1.7
2.5	−4.14	−4.95	−5.14
3.0	−1.63	−2.35	−2.88
3.5	−0.47	−0.87	−1.33
4.0	−0.10	−0.25	−0.48
4.5	−0.01	−0.05	−0.14

The results are clearly very sensitive to both X and the radius of the α-particle. If we take $X = 3.75\times10^{-13}$ cm so as to give the moment of inertia of C^{12} as determined by the first (4.44 MeV, 2+) excited level [11]) and $\varrho_\alpha = 1.4\times10^{-13}$ cm, then the non-additive exchange energy is certainly less than 0.5 MeV; this is small compared with the 7.3 MeV by which the α-particles are bound in C^{12}.

A convenient way to discuss the non-additive part of $V_X^{(1)}$ in the heavier α-nuclei is in terms of the ratio of the number t of triplets to the number b of bonds (i.e. the number of pairs of neighbouring α particles). The ratio (t/b) increases from 1/3 at C^{12} to 2/3 at O^{16}, and about 1 near the upper end of the chain of α-nuclei ($A = 40$). The non-additive energy therefore increases the effective binding energy per bond at the upper end of the chain by about $(2/3)\times0.47 = 0.3$ MeV over the value at the lower end. The magnitude of the variation is small compared with the observed binding energy per bond, which is about 2 MeV, and is therefore not large enough to destroy the additivity. The variation is actually of the order of magnitude of the observed deviation of the binding energy per bond from a constant value (ref. [1]), table 2.1, column 10); however, a steady increase of the binding energy per bond is not observed.

4. The Polarisation Interaction V_{POL}

A full treatment of V_{POL} is given in ref. [19]). For two α-particles, V_{POL} can, to a good approximation, be expressed in the form

$$V_{POL} \approx -\frac{\langle 0|(V_{\alpha\alpha}^2 - \langle 0|V_{\alpha\alpha}|0\rangle^2)|0\rangle}{\Delta E}, \tag{4.1}$$

where $V_{\alpha\alpha}$ is the interaction potential between the α-particles; (nucleon exchange is neglected); ΔE is a mean virtual excitation energy produced by the mutual distortion of the α-particles; the symbols $\langle 0| \ldots |0\rangle$ signify

the calculation of an expectation value with the product $\Psi_0^{(\alpha)}(1234)$ $\Psi_0^{(\alpha)}(5678)$ (see ref. [19]), eq. (2.24), (2.30)).

Formula (4.1) is convenient if the distortion which $V_{\alpha\alpha}$ produces is qualitatively of only one kind. This happens with the Rosenfeld mixture which leads to a correlation of the motion of the nucleons in the two α-particles in a way which depends on the spins and charges. However, with a potential which contains a direct part, there is in addition a distortion of the α-particles from the spherically symmetric shape. When there are both sorts of polarisation present, it is convenient to split up $V_{\alpha\alpha}$ into a direct part $V_{\alpha\alpha 1}$, and a spin-and-charge-dependent part $V_{\alpha\alpha 2}$. V_{POL} can then be expressed as the sum of two terms of the form (4.1) containing respectively $V_{\alpha\alpha 1}$ and $V_{\alpha\alpha 2}$, and each having its own excitation energy $(\Delta E)_1$ and $(\Delta E)_2$.

The results are given in ref. [19]) in tables 3.2 and 3.4. The calculations are based on a central Gaussian potential.

The treatment based on (4.1) applies to the case of two α-particles only. An outline of the extension of the discussion to more than two α-particles was given in ref. [8]), where it was shown that there is a non-additive polarisation energy if, and only if, the nuclear forces contain a direct part. The non-additive polarisation energy is estimated in § 4 of the accompanying paper for three α-particles and the Serber potential. It is negligible.

5. The Complete Interaction between two α-Particles

In this section we shall combine the results of the last three sections to compute the complete interaction energy of two α-particles. We shall then compare this interaction with some potentials with which attempts have been made to account for the properties of the C^{12} and O^{16} nuclei.

The results for V_{D}, $\langle V_{\mathrm{X}}^{(1)}\rangle$, V_{POL}, and the Coulomb energy have been collected together in figure 5.1; the data have been taken from tables 2.1, 3.1, and 3.2 of this paper, and tables 3.2 and 3.4 of ref. [19]); the Coulomb energy was calculated from the formula

$$E_{\mathrm{C}} = \frac{4e^2}{X}\,\mathrm{erf}\left(\frac{\sqrt{3}}{2}\,\frac{X}{\varrho_\alpha}\right), \tag{5.1}$$

which gives the Coulomb interaction energy of two distributions of charge of Gaussian shape, whose centres are separated by a distance X, and whose root mean square radii are ϱ_α; each distribution contains a total charge 2e. The curves shown are for $\varrho_\alpha = 1.4\times10^{-13}$ cm, except for the Coulomb energy where we have taken 1.6×10^{-13} cm so as to allow for the finite size of the charge distribution on the proton [17]).

The approximations which have been used in the calculation of $\langle V_{\mathrm{X}}^{(1)}\rangle$ and of V_{POL} are valid only if the overlap of the α-particles is not too large, and therefore break down when X is of the order of the diameter of an

α-particle or somewhat smaller, say somewhere between $X = 2$ and 3×10^{-13} cm. The explicit results are therefore drawn only for α-α spacings which are greater than 2.5×10^{-13} cm. At smaller spacings, V_X is expected to become strongly repulsive, but the velocity-dependent character of V_X

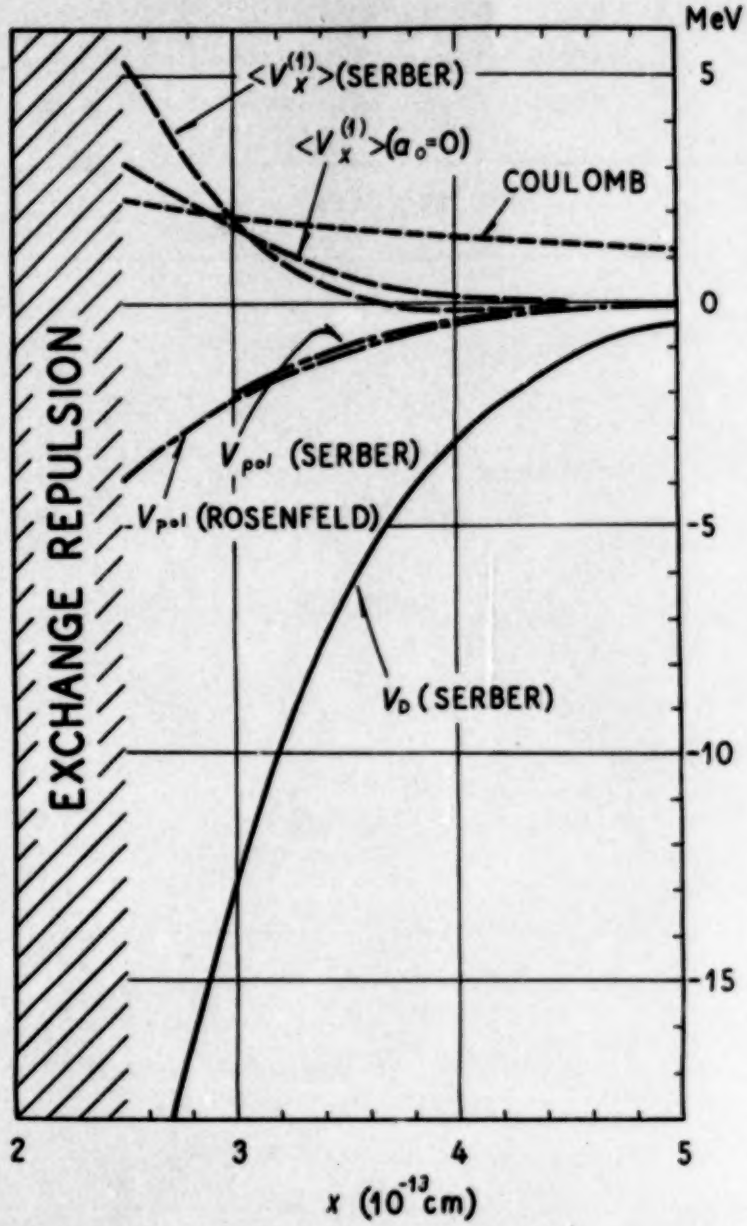

Fig. 5.1. Components of the α-α interaction energy.

Erratum. The labels on the two curves $\langle V_X^{(1)}\rangle$ should be interchanged.

does not allow us to represent this effect by a graph. We have therefore simply shaded the region $X < 2.5\times10^{-13}$ cm to indicate the presence of a repulsive interaction. In the region $X > 2.5\times10^{-13}$ cm, the use of $\langle V_X^{(1)}\rangle$ as a rough estimate of the probable effect of V_X is justified because we shall see that $\langle V_X^{(1)}\rangle$ is here only a small fraction of the complete inter-

action energy for the only set of potential parameters which shows promise of giving a satisfactory result.

One can see from figure 5.1 that for the Rosenfeld mixture, which has

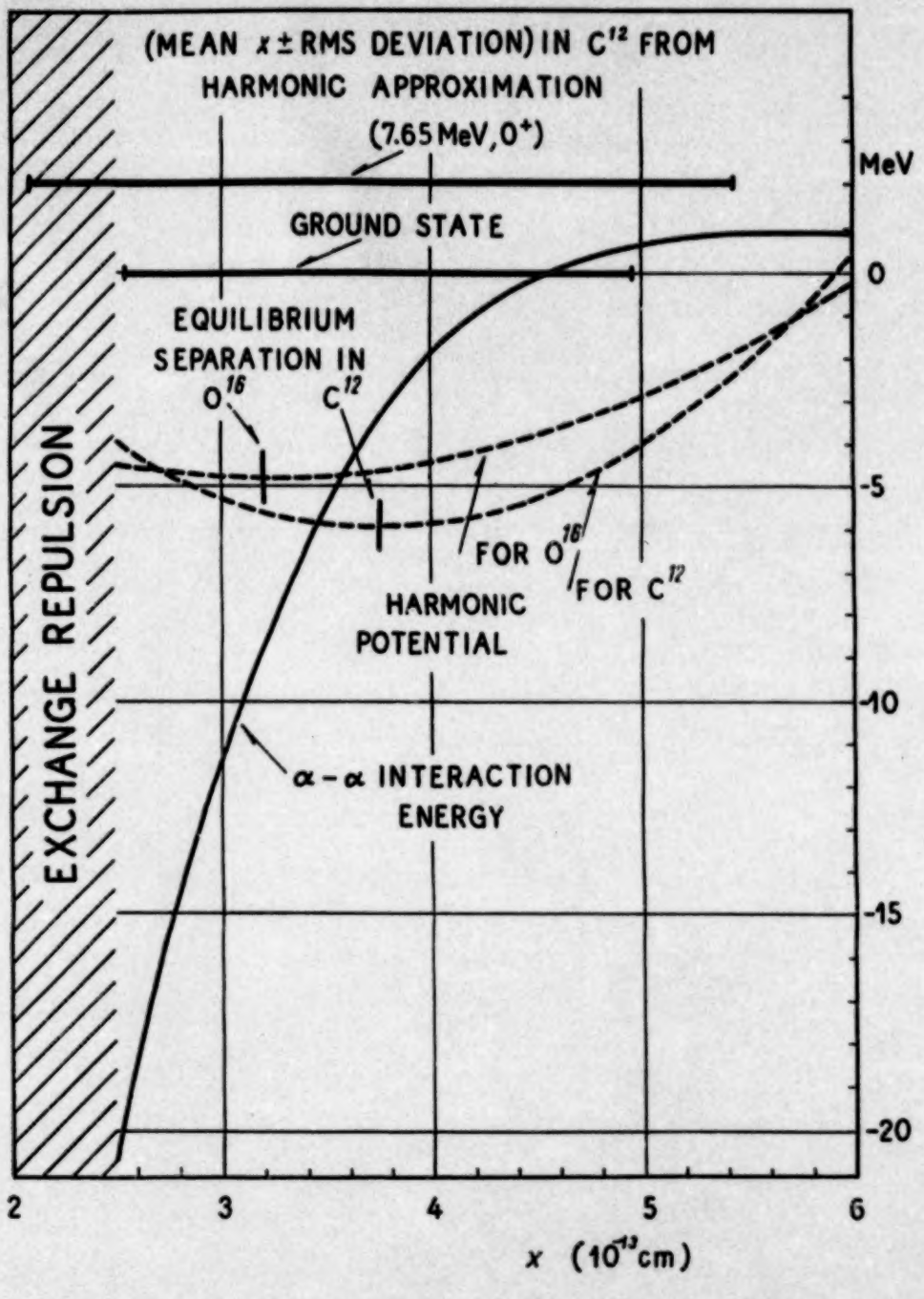

Fig. 5.2. Complete α-α interaction energy.

$a_0 = 0$, the complete nuclear interaction energy $(V_{\text{POL}} + \langle V_{\text{X}}^{(1)} \rangle)$ is smaller than the Coulomb energy so that this potential does not produce a satisfactory attraction between α-particles. For the Serber potential, which has

$a_0 \neq 0$ and therefore gives a substantial V_D component in the interaction energy, the complete interaction ($V_D+V_{POL}+\langle V_X^{(1)}\rangle$+Coulomb energy) has been plotted in figure 5.2. We have also drawn in the potentials which have been used in attempts to account for the energy levels of C^{12} and O^{16} on the α-particle model [11, 12]). The range of α-α-spacings occurring in C^{12} is also indicated by horizontal lines whose ends are at the points (mean α-α separation ± root mean square deviation); the figures are based on reference [11]). The potentials shown correspond to the sets of parameters labelled (*b*) in refs. [11]) and [12]) which correspond to nearly additive interactions; the small non-additive components of the interactions (*b*) obtained in refs. [11]) and [12]) have been neglected. The Gaussian potential was used for V_D; had we used the Yukawa shape, then the discrepancy between the calculated α-α interaction and the empirical harmonic potentials would have been much reduced.

6. Discussion

After our first assumption — that nuclear matter in a $4n$ nucleus is clustered into α-particles — we made the second assumption that overlapping α-particles repel one another. The origin of this repulsion has been ascribed in this paper to the exclusion principle, which expresses itself through the nucleon exchange energy V_X; in keeping with this idea, the single-pair-exchange part of this energy has been found to be repulsive. The assumption of the existence of a strong overlap repulsion implies that in the ground and not too highly excited states of nuclei consisting of α-particles, the probability of overlap is so small that the non-additive parts of V_X make only a negligible contribution to the energy.

If the assumption of the exchange repulsion is accepted, then the principal requirement of the α-particle model is an attraction which can operate when α-particles do not overlap, and which is additive. The α-particle must therefore have a 'fringing field', i.e. a field of force which extends outside the mass-distribution of the α-particle. There would appear to be two possible sources for a fringing field. The first, and the only one with forces which do not have a direct component ($a_0 = 0$), is due to a polarisation effect, i.e. Van der Waals forces. From fig. 5.1 it would seem that the range of V_{POL} is actually only slightly larger than that of the nucleon exchange forces and that therefore V_{POL} does not give a fringing field at all. However, the results shown in fig. 5.1 are deceptive on this point because of the way in which $\langle V_X^{(1)}\rangle$ has been calculated. The wave function which was used to calculate $\langle V_X^{(1)}\rangle$ implies a substantial oscillation of the mass-centres of the α-particles; as a consequence there is a fluctuation of the distance of separation of the mass-centres whose root mean square deviation is actually of the order of 10^{-13} cm and leads to an appreciable smearing out of the exchange

energy. In contrast, V_{POL} is a static potential and undergoes no such artificial increase of its range. When considering the additivity of a mixture of polarisation and nucleon exchange forces, one has to remember that when $a_0 = 0$, the single-pair-exchange tail of V_X is additive, so that the range of V_{POL} is not only larger than that of the whole of V_X, but very much larger than that of the non-additive part. However, it appears that the attraction provided by V_{POL} alone is too weak to overcome the Coulomb and nucleon exchange repulsions.

The second possible source of a fringing field around the α-particle, and therefore of an α-α attraction, lies in a direct component in the nuclear force (i.e. $a_0 < 0$). One can see from fig. 5.2 that the amount of direct interaction in the Serber potential gives an α-α attraction of the right order of magnitude. This attraction consists predominantly of V_D, but contains a component V_{POL} which approximately cancels the tail of the nucleon exchange repulsion.

The appeal to a direct component in the nuclear force to account for the attraction between α-particles leads to a number of difficulties. The first is that the single-pair-exchange tail of V_X is not additive because of the barrier-changing effect. Our estimate has shown the resultant non-additive contribution to the energy to be sufficiently small to be not inconsistent with an effectively additive α-α interaction. However, the sensitivity of the result to the α-α spacing is so strong that it is difficult to make a reliable estimate.

Another difficulty arises from the saturation conditions on the nuclear potential which require that the calculated binding energies of nuclei should be proportional to A rather than to A^2 and that the nuclear density should be independent of A. These conditions can be made to set a limit on a_0, and therefore on V_D, if one calculates the binding energy in the 'collapsed state' in which all nucleons are close enough to interact [23]) and demands that this binding energy should not exceed the observed value. (The collapsed state is the state in which all nucleons are within a sphere whose radius is half the range of the nuclear force.) The necessary formulae for a central nuclear potential of square well form acting between all pairs of nucleons are given by Blatt and Weisskopf [23]); we shall assume the potential to be independent of the spin and isobaric spin variables; tensor forces do not contribute because of the spherical symmetry. If one demands saturation for unlimited values of A, then one must have $a_0 \geqq 0$, irrespective of the potential strength and range. Therefore the demand for saturation in indefinitely heavy nuclei is inconsistent with an attractive V_D. To find the condition on a_0 implied by the more limited demand for saturation within only these values of A which are attained by nuclei known to exist, a particular potential has to be chosen; we take the square well with the

depth and range adjusted to fit the two-body singlet scattering length and effective range; using an effective range [24] of 2.57×10^{-13} cm and a scattering length [21]) of -23.69×10^{-13} cm one finds a potential strength of 15.6 MeV and range of 2.46×10^{-13} cm. Using the formulae of Blatt and Weisskopf, one finds that saturation at $A = 200$ requires $a_0 \geqq -0.44$, while at $A = 250$ one needs $a_0 \geqq -0.41$.

These results may be compared with the results of Frisch [25]) who investigated the effect of the exchange character of the potential on the stability of nuclear density. Frisch calculated binding energies for $A \approx 200$ with a central Yukawa potential and with nuclei of approximately normal density. He finds a tendency to collapse with the Serber exchange type $a_0 = -0.375 \cdot (1+q)/2$, and a rather stronger tendency to expand for the symmetric potential ($a_0 = 0$). (The criterion used there is the quantity $(1/A)(\partial E/\partial r_0)$, where r_0 is the usual radius parameter.) The demand for the stability of nuclear matter at the observed density therefore requires a rather weaker direct central force than there is in the Serber potential.

From both the results of the square well calculation for the collapsed state, and the result of Frisch's calculation just quoted, one is led to the conclusion that the limited demand for saturation within the known range of nuclei is compatible with an appreciable negative value of a_0 and an attractive direct interaction between α-particles.

A second and more important reason why the saturation difficulty is not crippling is that all that is required is an attractive direct tail in the nuclear force. If the saturation problem were to be resolved by the inclusion in the nuclear forces of a repulsive core [26]) of short radius while the nuclear forces retained a direct tail, then the α-particle would still have its fringing field, and the attraction between two α-particles would probably not be seriously affected. This argument is only qualitative because the model which has been considered here, based as it is on at most slightly distorted α-particles, cannot deal quantitatively with the correlations which the repulsive cores would require. However, it is unlikely that consideration of correlations would affect the conclusions because the most important feature of V_D is that it enables the α-particles to attract one another without seriously overlapping. (A repulsive core in the nuclear forces would actually help to provide the required short distance repulsion between α-particles.)

As one can see from fig. 5.1, the direct interaction V_D is numerically very much larger than $\langle V_X^{(1)} \rangle$ in the region of small overlap Since V_D is a static potential, it follows that in the region of small overlap the interaction energy of two α-particles can be represented by a static potential. In the region of a large overlap, which is shaded in figures 5.1 and 5.2, V_X plays an important role, and a potential representation of the interaction energy is not possible. However, if the assumption about the strongly

repulsive character of the α-α interaction in this region is correct, then it must be possible, at least in a first approximation, to represent V_X by an effective repulsive potential. Since the region of short separations is assumed to make a negligible contribution to the expectation value of the energy, the detailed form of the interaction energy cannot matter. The effective potential would be expected to rise steeply somewhere near the edge of the shaded region in figure 5.2. The complete interaction potential which one gets in this way is very different in all but general order of magnitude from the parabolic potentials, also indicated on figure 5.2, with which attempts have been made to account for the energy levels of C^{12} and O^{16}.

Neither of the two mechanisms which contribute to the α-α attraction gives rise to any important tendence for the α-particles to dissolve in nuclei with $A \lessapprox 40$. This is so for V_D because it has nothing to do with either nucleon exchange or polarisation, and for V_{POL} because of the estimates made in ref. [19]).

7. Conclusions

The attraction between α-particles which is required by the α-particle model can be accounted for as to order of magnitude if the force between two nucleons has a direct tail of the magnitude specified by the Serber potential. Nuclear forces which do not have a direct tail provide attractive Van der Waals forces, but these are too weak if the forces between nucleons are central. The interaction provided by a direct force is not inconsistent with additivity. Outside the region of substantial overlap, the interaction energy which one calculates from a direct component in the nuclear force can be represented by a static potential. The shape of this potential is very different from that of the parabolic potentials which have been used in attempts to account for the energy levels of C^{12} and O^{16} with the α-particle model. The mechanisms responsible for the attraction between α-particles give no more than a small probability for the α-particles to dissolve in nuclei with $A \lessapprox 40$.

I would like to thank Professor L. Rosenfeld for the interest he has taken in this problem. I am grateful to Professor R. Hofstadter for information before publication about the radius of the α-particle as determined by electron scattering, to Drs. Skyrme and Perring for a preprint of their paper on the relation of the α-particle and shell-models, and to Drs. Glassgold and Galonsky for a preprint of their paper on the α-particle model of C^{12}.

APPENDIX I

Further Remarks on Additivity

Margenau [4]) calculated the interaction energy of two α-particles with a wave function closely similar to that given in (3.13). The method used was

to calculate the expectation value of the total energy, and to subtract from this the energy for $|\mathbf{A}-\mathbf{B}| \to \infty$. The remainder was expressed as a function of $R \equiv |\mathbf{A}-\mathbf{B}|$. Margenau first calculated the interaction energy without polarisation, taking $a_0 = 0$. The result ($\Delta_1 E$ in Margenau's notation) was a positive energy falling monotonically with increasing distance from a maximum of 34 MeV at $R = 0$, and is roughly equivalent to what we have called V_X (see Margenau's figure 2). Margenau then argued that the only source of attraction possible was from Van der Waals forces which he approximated by the formula $\langle V^2 \rangle / E_m$, where $\langle V^2 \rangle$ is the expectation value of what we have called $V_{\alpha\alpha}$ and was calculated with neglect of nucleon exchange in the wave functions; E_m was a mean excitation energy which was not calculated but was assumed to be constant. To see whether the combination of $\Delta_1 E$ and Van der Waals forces would give an additive interaction, Margenau compared the ranges of $\langle V^2 \rangle$ and of the potential part of ($\Delta_1 E$) for pure Majorana forces. He writes as follows: "As an example, $\langle V^2 \rangle_{av}$ and $\overline{V}$ have been plotted for the case of pure Majorana forces in (Margenau's) figure 3, the ordinates having been so adjusted so as to make the two equal at $R = 0$. $\langle V^2 \rangle_{av}$, which corresponds to the Van der Waals force in molecular interactions, falls off much more rapidly than the exchange force $\overline{V}_{exch}$. To suppose that there exists a Van der Waals minimum beyond the range of the repulsive first-order forces would be fallacious.

This result has an important bearing on the additivity of the interaction between α-particles. In molecular problems and in nuclear problems as well, additivity holds for second-order forces, but not for first-order exchange forces. Inasmuch as there exists for molecules a range in R in which only second-order forces are present, additivity holds within this range. The absence of such a range in nuclear interactions destroys additivity completely". This conclusion is open to objection for the following reasons.

The Majorana potential which was used for the comparison of ranges has a substantial direct component which gives the corresponding $\Delta_1 E$ a long tail (due to what has been called V_D in this paper). However since this tail is additive, it is better to compare $\langle V^2 \rangle / E_m$ with the $\Delta_1 E$ which was calculated with $a_0 = 0$, and which, not having a direct component, has a much shorter range than the potential part of $\Delta_1 E$ calculated with the pure Majorana force (c.f. Margenau's figures 2 and 3). Moreover, the method used to estimate the polarisation force did not yield the components of longest range because Margenau wrote the space exchange operator in the form P_X; this is of course equivalent to $-P_\sigma P_\tau$ when the wave functions are properly anti-symmetrised, but when nucleon exchange is neglected, the notation P_X introduces overlap integrals which cut down the range below what one gets with $-P_\sigma P_\tau$. (From the equations in Margenau's Section 3 one can see that for the Majorana potential the rate of fall off of

$\langle V^2\rangle_{\text{av}}$ is governed mainly by the parameter which describes the α-particle wave function rather than by the one which describes the potential. In contrast, the $\langle V^2\rangle_{\text{av}}$ calculated for the pure Wigner potential, which contains no exchange operators, has its fall off determined mainly by the range of the potential). To compare ranges it is therefore best to compare $\langle V^2\rangle_{\text{av}}$ calculated with the pure Wigner potential with $\Delta_1 E$ for $a_0 = 0$; it turns out that these two have very nearly the same range.

To decide whether there can be a range of distances in which the interactions can or cannot be additive, there are now two cases to be considered.

(1) $a_0 = 0$. In this case the single-pair-exchange part of $\Delta_1 E$ is additive. If one picks out and calculates the single-pair-exchange approximation from Margenau's formulae, and subtracts the result from his $\Delta_1 E$, then the non-additive remainder falls to half its maximum (which occurs at $R = 0$) in about half the distance taken to do so by $\langle V^2\rangle_{\text{av}}$.

(2) $a_0 \neq 0$. There is now in the α-α interaction an additive V_{D} component which has a substantially longer range than the nucleon exchange energy.

In both cases (1) and (2) the additive interaction has a longer range than the non-additive interaction.

APPENDIX 2

Remarks on the Integrals in Equation (3.4)

To carry out the integrations denoted by $\int \mathrm{d}\mathbf{u}$ and $\int \mathrm{d}\mathbf{u}(\prod_{i=1}^{n} \mathrm{d}\mathbf{X}_i)$ in equation (3.4), one must first transform the particle position vectors $\mathbf{r}_i$ $(i = 1, \ldots 4n)$ into the new set of co-ordinates $(\mathbf{X}_1' \ldots \mathbf{X}_n'; \mathbf{u}_1 \ldots \mathbf{u}_{3n})$. We may suppose that these co-ordinates are linearly independent. Next, we wish to introduce the $\mathbf{X}_i$ explicitly as a part of the $\mathbf{u}_i$. A difficulty arises here because the $\mathbf{X}_i'$ and $\mathbf{X}_i$ are not linearly independent; they must, for example, satisfy the relation

$$\sum_{i=1}^{n} \mathbf{X}_i = \sum_{i=1}^{n} \mathbf{X}_i'. \qquad \text{(A.2.1)}$$

Therefore we cannot simply split off n of the $\mathbf{u}_i$ and express them in terms of the $\mathbf{X}_i$.

In a particular integral, let there be p linear relations between the $\mathbf{X}_i$ and $\mathbf{X}_i'$; let these relations be

$$f_q(\mathbf{X}_1' \ldots \mathbf{X}_n'; \mathbf{X}_1 \ldots \mathbf{X}_n) = 0, \qquad (q = 1 \ldots p). \qquad \text{(A.2.2)}$$

The conditions (A.2.2) still permit us to select $(n-p)$ linearly independent $\mathbf{X}_i$ in such a way that the remaining p co-ordinates $\mathbf{X}_i$ can be expressed in terms of the $(n-p)$ linearly independent $\mathbf{X}_i$ and the $\mathbf{X}_i'$. We now choose the $\mathbf{u}_i$ so that $(n-p)$ of them, which we denote by $\bar{\mathbf{u}}_i$, can be expressed in

terms of the $\mathbf{X}_i$ and $\mathbf{X}_i'$, while the remaining $\mathbf{u}_i$ $[i = (n-p+1, \ldots 3n)]$ are independent of the $\mathbf{X}_i$ and $\mathbf{X}_i'$.

The integrations in (3.4) are to be over all co-ordinates except the $\mathbf{X}_i'$. The integrals in H_D and V_{POL} are therefore straightforward, $(\int \mathrm{d}\mathbf{u})$ denoting $\int (\prod_{i=1}^{3n} \mathrm{d}\mathbf{u}_i)$. On the other hand in the exchange integrals in V_X the integration is

$$\int \left(\prod_{i=1}^{n-p} \mathrm{d}\bar{\mathbf{u}}_i \right) \cdot \left(\prod_{j=n-p+1}^{3n} \mathrm{d}\mathbf{u}_j \right) \ldots = \int \left(\prod_{j=n-p+1}^{3n} \mathrm{d}\mathbf{u}_j \right) \cdot J^3 \cdot \left(\prod_{i=1}^{n-p} \mathrm{d}\mathbf{X}_i \right) \ldots \quad \text{(A.2.3)}$$

where

$$J = \left| \frac{\partial \bar{\mathbf{u}}_i}{\partial \mathbf{X}_j} \right|. \quad \text{(A.2.4)}$$

We can extend the integration (A.2.3) over all the $\mathbf{X}_i$ if we insert a factor $\prod_{q=n-p+1}^{n} \delta[\mathbf{X}_q - \mathbf{X}_q(\mathbf{X}_1' \ldots \mathbf{X}_n'; \mathbf{X}_1 \ldots \mathbf{X}_{n-p})]$ where the formulae

$$\mathbf{X}_q - \mathbf{X}_q(\mathbf{X}_1' \ldots \mathbf{X}_n'; \mathbf{X}_1 \ldots \mathbf{X}_{n-p}) = 0, \qquad (q = n-p+1, \ldots n) \quad \text{(A.2.5)}$$

are the expressions of the last p of the $\mathbf{X}_i$ in terms of the other $\mathbf{X}_i$ and the $\mathbf{X}_i'$.

The exchange integrals finally become, in place of (A.2.3),

$$\int \left[\left(\prod_{j=n-p+1}^{3n} \mathrm{d}\mathbf{u}_j \right) \cdot J^3 \cdot \prod_{q=n-p+1}^{n} \delta(\mathbf{X}_q - \mathbf{X}_q(\mathbf{X}_1' \ldots \mathbf{X}_n'; \mathbf{X}_1 \ldots \mathbf{X}_{n-p}) \right] \left(\prod_{i=1}^{n} \mathrm{d}\mathbf{X}_i \right) \ldots \quad \text{(A.2.6)}$$

The contents of the square bracket in (A.2.6) are what is denoted by $\mathbf{du}$ in the integrals in V_X in (3.4).

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INELASTIC HIGH ENERGY NUCLEON SCATTERING ON LIGHT NUCLEI

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Abstract: The excitation of a nucleus bombarded by nucleons of a few hundreds of MeV is considered as the result of individual interactions of the incoming nucleon with the nucleons of the nucleus. A simple two-particle model, in which a bombarding particle excites a particle moving in a potential well, has been calculated in the first Born approximation. It is shown that the resulting angular distributions have a strong forward dip and vanish at large angles.

To a certain extent this result still holds for the case of a nucleus consisting of Fermi-particles with common and isobaric spins. In first order only those states can be excited whose wave functions have, in the sense of the independent particle model, a component differing only in the state of one particle from the ground state wave function.

In the case of neutron scattering the cross section has a forward dip if the space part of the wave function of the excited particle is orthogonal to the space part of its ground state wave function. If this is not so, for example if the excitation can occur by a pure spin exchange, there may be a forward maximum.

In proton scattering the Coulomb interaction could result in a forward peaked angular distribution by dipole excitation as well as in the above mentioned case of a monopole transition.

In higher approximation two or more particles in the nucleus can be simultaneously excited. The resulting cross sections are expected to be smaller and more isotropic than in the case of a one-particle excitation.

The angular distributions for two types of *collective* excitation have also been calculated, in the first Born approximation, namely for the lowest quadrupole surface vibration and for the dipole vibration of the nuclear charge.

The qualitative features of the experimentally measured cross sections of the scattering of 185 MeV protons on C^{12} are shown to be in agreement with the theoretical expectations.

1. Introduction

The treatment of low energy scattering of nucleons on nuclei is simplified by the facts that often a compound nucleus is formed and that only a relatively small number of angular momentum eigenfunctions of the incoming nucleon participate in the collision. In such a case the scattered particles and reaction products may have a simple angular distribution resulting from the interference of a few angular momentum eigenfunctions. From such distributions it is then often possible to draw conclusions concerning spins and parities of the nuclear states involved in the scattering process.

Tnis situation is completely changed at bombarding energies of the order of hundreds of MeV. In this case no compound nucleus is formed and the

number of angular momentum eigenfunctions involved in the collision of a nucleon with a certain nucleus, being roughly proportional to the root of the bombarding energy, becomes large. In the scattering of 200 MeV protons even on such a light nucleus as carbon this number is at least 10 and consequently the resulting angular distribution no longer show the simple features of the low energy ones.

In the scattering at high energies it would seem, therefore, that the analysis of the incoming and scattered waves into components of definite angular momentum may not be the simplest procedure to adopt. In the following we use a decomposition in linear momentum eigenfunctions.

Clearly at high energies the experimentally observed crosssections will no longer allow one to make direct conclusions about nuclear spin values, but, as we hope to show, may give other valuable information concerning the character of the excited states.

In the high energy scattering of nucleons on light nuclei a compensation for the complications resulting from the many participating angular momenta and for the fact that the idea of a compound nucleus formation is no longer applicable, comes in through the circumstance that one may approximately consider the collisions as individual interactions between the incoming nucleon and the nucleons of the bombarded nucleus [1]). In addition, the first Born approximation is expected to be better.

In the first part of the present article we deduce some qualitative properties of the angular distributions which follow from the above assumptions. The treatment followed, and many of results obtained, are very similar to the ones of references [2]), and [3]), which concern somewhat different problems.

In the last part the possiblity of two types of collective excitations is briefly considered and finally some remarks are made concerning the interpretation of preliminary measurements on the inelastic scattering of 185 MeV protons by C^{12}.

2. Calculation with a Simple Model

We consider the two-body interactions of the bombarding nucleon with the nucleons of the bombarded nucleus as the perturbation in a Born approximation. In first order, only those nuclear states can be excited whose wavefunctions have, in the sense of the independent particle model, a component differing in the state of only one particle from the nuclear ground state.

It seems plausible that some typical features of such a transition should already appear in a simple two-particle model in which a bombarding particle excites a particle moving in a potential well. We shall later show this assumption to be justified.

The following model is chosen: a spinless particle moves in a potential

well and another spinless particle, of equal mass but assumed distinguishable from the first one, is scattered on it. We shall calculate the angular distribution of the scattered particle corresponding to a certain excitation of the bound particle.

In this part the following notation will be used:

$\psi_i(\mathbf{r}_1, \mathbf{r}_2)$ — initial
$\psi_f(\mathbf{r}_1, \mathbf{r}_2)$ — final
} normalized wave function of the two-body system,

$\varphi_i(\mathbf{r}_2)$ — initial
$\varphi_f(\mathbf{r}_2)$ — final
} normalized wave function of the particle in the potential well,

$\hbar\mathbf{k}$ — initial
$\hbar\mathbf{k}'$ — final
} momentum of the scattered particle,

L^3 — volume of the box in which we normalize the wave functions,

$V(r_2)$ — potential well,

$V_{12}(\mathbf{r}_2 - \mathbf{r}_1)$ — interaction operator between the particles.

The total Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\Delta_1 - \frac{\hbar^2}{2m}\Delta_2 + V_{12}(\mathbf{r}_2 - \mathbf{r}_1) + V(r_2). \tag{1}$$

For the unperturbed Hamiltonian we take

$$H_0 = -\frac{\hbar^2}{2m}\Delta_1 - \frac{\hbar^2}{2m}\Delta_2 + V(r_2). \tag{2}$$

The perturbation is

$$H' = V_{12}. \tag{3}$$

We have

$$\psi_i = L^{-3/2} e^{i\mathbf{k}\cdot\mathbf{r}_1}\varphi_i(\mathbf{r}_2); \qquad \psi_f = L^{-3/2} e^{i\mathbf{k}\cdot\mathbf{r}_1}\varphi_f(\mathbf{r}_2). \tag{4}$$

The cross section is

$$\sigma(\mathbf{k}, \mathbf{k}') = \frac{k'}{k}\left(\frac{m}{2\pi\hbar^2}\right)^2 L^6 |H'_{fi}|^2, \tag{5}$$

where

$$H'_{fi} = L^{-3}\int e^{-i\mathbf{k}'\cdot\mathbf{r}_1}\varphi_f^*(\mathbf{r}_2) V_{12}\, e^{i\mathbf{k}\cdot\mathbf{r}_1}\varphi_i(\mathbf{r}_2) d\mathbf{r}_1\, d\mathbf{r}_2. \tag{6}$$

We analyse φ_i and φ_f in momentum eigenfunctions:

$$\varphi_i(\mathbf{r}_2) = L^{-3/2}\sum_{\mathbf{k}_i}[g_i(\mathbf{k}_i)\, e^{i\mathbf{k}_i\cdot\mathbf{r}_2}],$$

$$\varphi_f(\mathbf{r}_2) = L^{-3/2}\sum_{\mathbf{k}_f}[g_f(\mathbf{k}_f)\, e^{i\mathbf{k}_f\cdot\mathbf{r}_2}], \tag{7}$$

where

$$g_i(\mathbf{k}_i) = L^{-3/2}\int \varphi_i(\mathbf{r}_2)\, e^{-i\mathbf{k}_i\cdot\mathbf{r}_2}\, d\mathbf{r}_2, \tag{8}$$

$$g_f(\mathbf{k}_f) = L^{-3/2}\int \varphi_f(\mathbf{r}_2)\, e^{-i\mathbf{k}_f\cdot\mathbf{r}_2}\, d\mathbf{r}_2.$$

Substituting (7) into (6), we get

$$H'_{fi} = L^{-6} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_f} [g_f^*(\mathbf{k}_f) g_1(\mathbf{k}_1) \int \mathrm{e}^{-i\mathbf{k}_f \cdot \mathbf{r}_2 - i\mathbf{k}' \cdot \mathbf{r}_1} V_{12} \, \mathrm{e}^{i\mathbf{k}_1 \cdot \mathbf{r}_2 + i\mathbf{k} \cdot \mathbf{r}_1} \, \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2]. \quad (9)$$

The integral can be simplified by introducing relative and centre of mass coordinates. We assume that V_{12} is a multiplicative operator. Then

$$\begin{aligned} &\int \mathrm{e}^{-i\mathbf{k}_f \cdot \mathbf{r}_2 - i\mathbf{k}' \cdot \mathbf{r}_1} V_{12}(\mathbf{r}_2 - \mathbf{r}_1) \, \mathrm{e}^{i\mathbf{k}_1 \cdot \mathbf{r}_2 + i\mathbf{k} \cdot \mathbf{r}_1} \, \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \\ &= L^3 \delta(\mathbf{k}_1 + \mathbf{k} - \mathbf{k}_f - \mathbf{k}') \int \mathrm{e}^{-i\mathbf{k}' \cdot \mathbf{r}} V_{12}(\mathbf{r}) \, \mathrm{e}^{i\mathbf{k} \cdot \mathbf{r}} \, \mathrm{d}\mathbf{r}; \end{aligned} \quad (10)$$

eq. (10) and (9) give

$$H'_{fi} = L^{-3} \int \mathrm{e}^{-i\mathbf{k}' \cdot \mathbf{r}} V_{12}(r) \, \mathrm{e}^{i\mathbf{k} \cdot \mathbf{r}} \, \mathrm{d}\mathbf{r} \sum_{\mathbf{k}_f} [g_f^*(\mathbf{k}_f) g_1(\mathbf{k}_f + \mathbf{k}' - \mathbf{k})]. \quad (11)$$

In this product the integral is a matrix element for the scattering of free particles 1 and 2 and depends on V_{12} and $\mathbf{k}' - \mathbf{k}$, which is $1/\hbar$ times the momentum change of the scattered particle. The sum depends on the properties of the final and initial states and on $\mathbf{k}' - \mathbf{k}$, but not on V_{12}. In the following this last term will be referred to as the "nuclear factor".

In order to get a qualitative idea of the angular dependence of H'_{fi} we first connect the integral in (11) with a cross section for free scattering. Consider the following hypothetical scattering experiment: a beam of particles 1 of momentum $\hbar\mathbf{k}$ is scattered on a beam of particles 2 of a certain initial momentum $\hbar\mathbf{k}_1'$ and final momentum $\hbar\mathbf{k}_f'$. The differential cross section for the scattering of particle 1 into a state with momentum $\hbar\mathbf{k}'$ is, in first Born approximation

$$\begin{aligned} \sigma_{fi}(\mathbf{k}, \mathbf{k}') &= \frac{k'}{k} \left(\frac{m}{2\pi\hbar^2} \right)^2 L^{-6} \left| \int \mathrm{e}^{-i\mathbf{k}_f' \cdot \mathbf{r}_2 - i\mathbf{k}' \cdot \mathbf{r}_1} V_{12} \, \mathrm{e}^{i\mathbf{k}_1' \cdot \mathbf{r}_2 + i\mathbf{k} \cdot \mathbf{r}_1} \, \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \right|^2 \\ &= \frac{k'}{k} \left(\frac{m}{2\hbar^2} \right) L^{-3} \delta(\mathbf{k}_1' + \mathbf{k} - \mathbf{k}_f' - \mathbf{k}') \left| \int \mathrm{e}^{-i\mathbf{k}' \cdot \mathbf{r}} V_{12}(r) \, \mathrm{e}^{i\mathbf{k} \cdot \mathbf{r}} \, \mathrm{d}\mathbf{r} \right|^2. \end{aligned} \quad (12)$$

Because of momentum and energy conservation there is for fixed $\mathbf{k}$ and $\mathbf{k}'$ one condition imposed on $\mathbf{k}_1'$ to make this process possible, but otherwise this quantity might, in principle, be chosen arbitrarily; $\mathbf{k}_f'$ is then determined by $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{k}_1'$. $\sigma_{fi}(\mathbf{k}_f \mathbf{k}')$ is in the present approximation dependent only on $\mathbf{k}' - \mathbf{k}$.

In equation (9) $\hbar\mathbf{k}_1$ and $\hbar\mathbf{k}_f$ have because of (7) the physical meaning of internal momenta of the bound particle in the initial and final states. In general $\mathbf{k}_1'$ and $\mathbf{k}_f'$ in (12) cannot be taken exactly equal to a set of values of $\mathbf{k}_1$ and $\mathbf{k}_f$ because the energy conservation means a somewhat different condition in the expressions (9) and (12), but it is obvious from physical point of view that there exist admissible values of $\mathbf{k}_1'$ and $\mathbf{k}_f'$ lying

in the range of $\mathbf{k}_1$ and $\mathbf{k}_f$. In the nuclear case these momenta correspond to a kinetic energy of the order of 20 MeV, which is about one third of the momentum corresponding to a bombarding energy of 200 MeV. Hence the angular distribution of $\sigma_{tr}(\mathbf{k}, \mathbf{k}')$ will not be very different from the usual cross section for which $\mathbf{k}_1' = 0$. From (11), (12), and (5) the inelastic cross section becomes

$$\sigma(\mathbf{k}, \mathbf{k}') = \sigma_{tr}(\mathbf{k}, \mathbf{k}') \left|\sum_{\mathbf{k}_f} g_f^*(\mathbf{k}_f) g_1(\mathbf{k}_f + \mathbf{k}' - \mathbf{k})\right|^2.$$

Passing to the limit $L \to \infty$ and using instead of the box normalization the delta-function normalization one finds

$$\sigma(\mathbf{k}, \mathbf{k}') = \sigma_{tr}(\mathbf{k}, \mathbf{k}') \left|\int g_f^*(\mathbf{k}_f) g_1(\mathbf{k}_f + \mathbf{k}' - \mathbf{k})\, d\mathbf{k}_f\right|^2, \tag{13}$$

in which in place of (8) we have

$$\begin{aligned} g_1(\mathbf{k}) &= (2\pi)^{-3/2} \int \varphi_1(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}, \\ g_f(\mathbf{k}) &= (2\pi)^{-3/2} \int \varphi_f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}. \end{aligned} \tag{14}$$

For the special case that the nuclear potential well is taken to be zero and we choose for the initial and final wave functions of particle 2 the momentum eigenfunctions with the eigenvalues $\hbar\mathbf{k}''$ and $\hbar\mathbf{k}'''$, we find

$$\begin{aligned} \sigma(\mathbf{k}, \mathbf{k}') &= \sigma_{tr}(\mathbf{k}, \mathbf{k}') \int \delta(\mathbf{k}_f - \mathbf{k}''') \delta(\mathbf{k}_f + \mathbf{k}' - \mathbf{k} - \mathbf{k}'') d\mathbf{k}_f \\ &= \sigma_{tr}(\mathbf{k}, \mathbf{k}') \delta(\mathbf{k}''' + \mathbf{k}' - \mathbf{k} - \mathbf{k}'') \end{aligned}$$

as expected.

As already remarked, $\sigma_{tr}(\mathbf{k}, \mathbf{k}')$ has an angular dependence similar to that of the usual cross section for particle 1 with momentum $\hbar\mathbf{k}$ scattered on particle 2 at rest, at least in the case of interest, in which the momentum of the bombarding particle is considerably larger than the average momentum of the bound particle. The cross section will, therefore, have the qualitative appearance of the usual particle 1-particle 2 cross section modified by the square of the nuclear factor

$$\left|\int |g_f^*(\mathbf{k}_f) g_1(\mathbf{k}_f + \mathbf{k}' - \mathbf{k}) d\mathbf{k}_f\right|^2. \tag{15}$$

The dependence of this factor on the momentum change, that is on the scattering angle, of particle 1 has some general and simple features. One may observe that the integral is the scalar product in momentum space of the final and initial wave functions, with the latter function translated by the vector $\mathbf{k}-\mathbf{k}'$. This vector equals $-1/\hbar$ times the momentum change suffered by the bombarding particle as a result of the collision. From this it follows that

a) For $\mathbf{k}'-\mathbf{k}$ small, that is for scattering in the forward direction †, the integral tends to zero because of the orthogonality of g_f and g_i.

b) For $\mathbf{k}'-\mathbf{k}$ larger than the sum of the radii of the momentum distributions of particle 2 in the initial and final states the functions in the integral do not overlap and the integral vanishes again.

The nuclear factors will only have an appreciable magnitude between the limits a) and b). These general features may be illustrated by the examples of the excitations of particle 2 from the 0S level into the 2S, 4S and 1P levels of a parabolic potential (the numbers in the level assignments represent the total number of oscillator quanta)

$$V(r_2) = \tfrac{1}{2}m\omega^2 r_2^2.$$

For the nuclear factor in the 0S → 2S transition one finds

$$\int g^*_{2S}(\mathbf{k}_f)g_{0S}(\mathbf{k}_f+\mathbf{k}'-\mathbf{k})d\mathbf{k}_f = 0.204\,\frac{\hbar}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{\hbar\omega}\exp\left[-\frac{\hbar^2}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{2\hbar\omega}\right].$$

The nuclear factor for the 0S → 4S excitation is

$$\int g^*_{4S}(\mathbf{k}_f)g_{0S}(\mathbf{k}_f+\mathbf{k}'-\mathbf{k})d\mathbf{k}_f = 0.0454\left[\frac{\hbar^2}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{\hbar\omega}\right]^2\exp\left[\frac{-\hbar^2}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{2\hbar\omega}\right].$$

The sum of the squares of the nuclear factors for the three 0S → 1P transitions is

$$\sum\left|\int g^*_{1P}(\mathbf{k}_f)g_{0S}(\mathbf{k}_f+\mathbf{k}'-\mathbf{k})d\mathbf{k}_f\right|^2 = \left[\frac{\hbar^2}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{\hbar\omega}\right]\exp\left[\frac{-\hbar^2}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{\hbar\omega}\right].$$

For later use we also write down the nuclear factor for the 0S → 0S transition, which corresponds to the elastic scattering,

$$\int g^*_{0S}(\mathbf{k}_f)g_{0S}(\mathbf{k}_f+\mathbf{k}'-\mathbf{k})d\mathbf{k}_f = \exp\left[-\frac{\hbar^2}{2m}\,\frac{(\mathbf{k}'-\mathbf{k})^2}{2\hbar\omega}\right].$$

Because the final and initial wave functions are identical in this case there exists a maximum nearly equal to 1 in the forward direction.

In fig. 1 we have plotted the squares of these factors; $\hbar\omega$ was taken to be 18 MeV, which corresponds to the potential well of the C^{12} nucleus, as obtained from the expression [4])

$$\hbar\omega = 41A^{-1/3}\ \text{MeV}.$$

We took 185 MeV as bombarding energy $(\hbar^2/2m)k^2$.

† For a bombarding energy of 200 MeV, an excitation of 20 MeV and an average internal momentum of 20 MeV, the momentum change of an inelastically scattered particle in the forward direction is $(\sqrt{200}-\sqrt{180})/\sqrt{20} = 15\ \%$ of the internal momentum. In the forward direction the square of the nuclear factor can be shown to contain no power lower than the second power of that fraction.

The angular distribution of the scattered particles is found by multiplying the square of the nuclear factor by the free particle cross section (see eq. (13)). As long as this cross section does not show a very singular be haviour

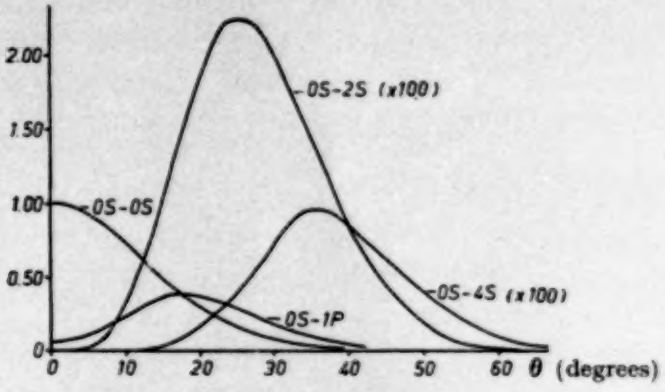

Fig. 1. The square of the nuclear factors for the 0S → 0S, 0S → 2S, 0S → 4S and 0S → 1P excitations of the simple model as defined in the text.

in its angular dependence, the general shape of the angular distribution should have the same character as the plotted nuclear factors, although the position of the maximum in the inelastic cross sections may be considerable shifted. We show this in fig. 2 somewhat more explicitely for the

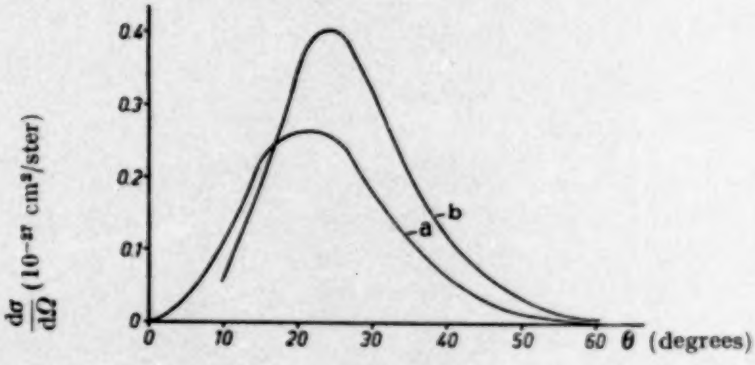

Fig. 2. The square of the 0S → 2S nuclear factor of fig. 1 multiplied by the 200 MeV p-n cross section in the laboratory system (curve a) and by the 200 MeV p-p cross section (curve b). The figure shows that the qualitative appearance of such product functions is mainly determined by the nuclear factor and not very sensitive to variations in the cross section.

0S → 2S excitation by taking for $\sigma_{tr}(\mathbf{k}, \mathbf{k}')$ in (13) the experimental (p-n) and (p-p) cross sections at 200 MeV.

The matrix element for the Coulomb interaction is proportional to $(\mathbf{k}'-\mathbf{k})^{-2}$. If V_{12} contains the Coulomb interaction, appreciable cross sections in the forward direction could result in a case in which the nuclear factor at small angles is proportional to a power of $(\mathbf{k}'-\mathbf{k})$ less than two. It is easy to show that this can only be so for the elastic scattering and for a dipole transition, which occurs between two states belonging to two

neighbouring levels in the oscillator potential, for example for the 1P → 2D and 1P → 2S transitions. For the light nuclei, which are here under consideration, the latter excitations lead just to levels of the "giant resonances"[5]) and we shall later estimate the order of magnitude of the sum of their cross sections by a collective model. In general the Coulomb cross sections at high energies are expected to be negligible at all except small angles.

3. The General Case

Until now we have considered a rather unrealistic model of only two distinguishable particles. It is, however, not difficult to see that our results to a certain extent still hold for the general case of several Fermi-particles with common and isobaric spins.

Because of the uncertainties in the nucleon interaction operators, in the accuracy of the approximation and in the nuclear wave functions, the treatment has not yet been carried through further than is necessary to show the connection with the results obtained in the two-particle model.

In the following we first introduce spin and isobaric spin coordinates, but still consider the bombarding particle as distinguishable from the nuclear particles which are taken to be identical. By summing over the matrix elements whose final wave functions after antisymmetrization lead to the same excited state, one obtains the corresponding matrix element for the case that all particles are indistinguishable [6]).

Consider a high energy proton which is scattered on a nucleus consisting of A nucleons. Assuming two-body interactions, the total Hamiltonian is

$$H = \sum_{1}^{A+1} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{1}^{A+1} \sum_{1}^{A+1} V_{ik}. \tag{1'}$$

For the unperturbed Hamiltonian we take

$$H_0 = \sum_{1}^{A+1} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{2}^{A+1} \sum_{2}^{A+1} V_{ik}. \tag{2'}$$

The perturbation is

$$H' = \sum_{2}^{A+1} V_{1k}. \tag{3'}$$

The following notation is used:

φ_1 — initial antisymmetric nuclear wave function,
φ_f — final antisymmetric nuclear wave function,
s_l (l is 1 or 2) — the usual one-particle spin eigenfunctions,
t_j (j is 1 or 2) — the isobaric spin eigenfunctions,
$\hbar\mathbf{k}$ — the initial momentum of the scattered particle,
$\hbar\mathbf{k}'$ — the final momentum of the scattered particle.

The initial wave function is

$$L^{-3/2}\, e^{i\mathbf{k}\cdot\mathbf{r}_1}\, s_1(1)\, t_2(1)\, \varphi_1(2, 3 \ldots, A+1).$$

The unsymmetrized wave functions, which after antisymmetrization lead to the final state, are

$$L^{-3/2}\, e^{i\mathbf{k}'\cdot\mathbf{r}_1} s_f(1)\, t_2(1)\, \varphi_f(2, 3, \ldots, A+1),$$
$$L^{-3/2}\, e^{i\mathbf{k}'\cdot\mathbf{r}_2} s_f(2)\, t_2(2) \quad \varphi_f(1, 3, \ldots, A+1),$$
$$L^{-3/2}\, e^{i\mathbf{k}'\cdot\mathbf{r}_{A+1}} s_f(A+1)\, t_2(A+1)\, \varphi_f(1, 2, \ldots A).$$

We indicate the antisymmetrical sum over these functions by putting two brackets around the first one. The sum of the matrix elements between the final states and the ground state is

$$H'_{fi} = L^{-3} \langle [e^{i\mathbf{k}'\cdot\mathbf{r}_1} s_f(1)\, t_2(1)\, \varphi_f(2, 3, \ldots A+1)]| \sum_2^{A+1} V_{1k}|\, e^{i\mathbf{k}\cdot\mathbf{r}_1} s_1(1)\, t_2(1)\, \varphi_1(2 \ldots A+1)\rangle.$$

Because of the antisymmetry of φ_f and φ_1 in the particles $2, \ldots A+1$, one may write

$$H'_{fi} = AL^{-3} \langle [e^{i\mathbf{k}'\cdot\mathbf{r}_1} s_f(1)\, t_2(1)\, \varphi_f(2 \ldots A+1)]| V_{12}|\, e^{i\mathbf{k}\cdot\mathbf{r}_1} s_1(1)\, t_2(1)\, \varphi_1(2 \ldots A+1)\rangle.$$

In the sum indicated by the brackets only the two summands in which particles 1 or 2 appear in the exponential contribute significantly. Mathematically this follows from the fact that the rapidly oscillating function $\exp(i\mathbf{k}'\cdot\mathbf{r}_i)$ is practically orthogonal to the nuclear wave function with respect to integration over $\mathbf{r}_i$, and physically it is also clear that a nucleon can only receive the very large momentum $\hbar\mathbf{k}'$ if it interacts directly with a fast particle.

We symbolize the antisymmetrization in particles 1 and 2 and the normalization of a wave function by bars, and we may then write

$$H'_{fi} = AL^{-3} \langle |\, e^{i\mathbf{k}'\cdot\mathbf{r}_1} s_f(1)\, t_2(1)\, \varphi_f(2, 3, \ldots A+1)| V_{12}\, |e^{i\mathbf{k}\cdot\mathbf{r}_1} s_1(1)\, t_2(1)\, \varphi_1(2 \ldots A+1)|\rangle. \qquad (6')$$

We analyse the nuclear wave functions in the complete eigenfunction system of particle 2,

$$L^{-3/2}\, e^{i\mathbf{k}\cdot\mathbf{r}_2}\, s_m(2)\, t_2(2),$$

as follows;

$$\varphi_1(2 \ldots A+1) = L^{-3/2} \sum_{\mathbf{k}_1} \sum_m \sum_s g_1(\mathbf{k}_1, m, s, 3 \ldots A+1)\, s_m(2)\, t_s(2)\, e^{i\mathbf{k}_1\cdot\mathbf{r}_2},$$
$$\varphi_f(2 \ldots A+1) = L^{-3/2} \sum_{\mathbf{k}_f} \sum_n \sum_t g_f(\mathbf{k}_f, n, t, 3 \ldots A+1)\, s_n(2)\, t_t(2)\, e^{i\mathbf{k}_f\cdot\mathbf{r}_2}. \qquad (7')$$

The functions g are still antisymmetric in all particles $3 \ldots A+1$. Inserting (7′) in (6′) results in

$$H'_{fi} = AL^{-3} \sum_{\mathbf{k}_1} \sum_{m} \sum_{s} \sum_{\mathbf{k}_f} \sum_{n} \sum_{t} \langle |e^{i\mathbf{k}'\cdot\mathbf{r}_1} s_f(1)\, t_2(1)\, e^{i\mathbf{k}_f\cdot\mathbf{r}_2} s_n(2)\, t_t(2)| \\ V_{12}\, |e^{i\mathbf{k}\cdot\mathbf{r}_1} s_1(1)\, t_2(1)\, e^{i\mathbf{k}_i\cdot\mathbf{r}_2} s_m(2)\, t_s(2)| \rangle \\ \times \int g_f^*(\mathbf{k}_f, n, t, 3 \ldots A+1)\, g_1(\mathbf{k}_1, m, s, 3 \ldots A+1)\, \mathrm{d}(3) \ldots \mathrm{d}(A+1) \\ = \sum_{m} \sum_{n} \sum_{s} e^{i\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)} s_f(1)\, t_2(1)\, s_n(2)\, t_s(2)| \qquad (9') \\ V_{12}\, |e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} s_1(1)\, t_2(1)\, s_m(2)\, t_s(2)| \rangle \\ \times \int g_f^*(\mathbf{k}_f, n, s, 3 \ldots A+1) g_1(\mathbf{k}_f+\mathbf{k}'-\mathbf{k}, m, s, 3 \ldots A+1) \mathrm{d}\mathbf{k}_f \mathrm{d}(3) \ldots \mathrm{d}(A+1).$$

We have introduced the delta-function normalization for the g's and put $\mathbf{k}_1 = \mathbf{k}_f + \mathbf{k}' - \mathbf{k}$ and $t = s$, using the momentum and charge conservation property of the operator V_{ik} and assuming that it acts on the space coordinates as a multiplicative operator. This is the case for almost all of the interaction operators used in practice.

The expression (9′) is comparable with (9). It is, however, no longer possible to get the analogue of (13) because the scattering amplitudes between states of different spin and isobaric spin orientations interfere. Each of the scattering amplitudes has a different nuclear factor which, with delta-function normalization, reads

$$\int g_f^*(\mathbf{k}_f, n, t, 3 \ldots A+1) g_1(\mathbf{k}_f+\mathbf{k}'-\mathbf{k}, m, t, 3 \ldots A+1) \mathrm{d}\mathbf{k}_f \mathrm{d}(3) \ldots \mathrm{d}(A+1). \qquad (15')$$

These factors do not necessarily vanish in the forward direction ($\mathbf{k}'-\mathbf{k} \approx 0$), because the orthogonality of the final and initial states results in a somewhat different condition:

$$\sum_{t} \sum_{n} \int g_f^*(\mathbf{k}_f, n, t, 3 \ldots A+1) g_1(\mathbf{k}_f, n, t, 3 \ldots A+1) \mathrm{d}\mathbf{k}_f \mathrm{d}(3) \ldots \mathrm{d}(A+1) = 0.$$

But for many of the single particle levels it can be shown that the space part of the wave function of the excited particle already takes care of the orthogonality. In this case the nuclear factors of all summands vanish in the forward direction and consequently also the cross section. For the same reason as in the case treated earlier this is always true for sufficiently large angles where $\hbar|\mathbf{k}'-\mathbf{k}|$ is greater than the sum of the radii of the momentum distributions of the final and initial states.

In the forward direction ($\mathbf{k}'-\mathbf{k} \approx 0$) the nuclear factor (15′) equals the scalar product of the initial and final nuclear wavefunctions in which the integration is performed over all $3A$ space-coordinates and the summation over all spin-variables except over the two spin-variables of one particle. To investigate the value of this scalar product it is convenient to

expand the wave functions in the complete system of antisymmetrized eigenfunctions of the total orbital angular momentum L, the total spin angular momentum S and the total isobaric spin T, as is also usual in pure L-S-coupling considerations (7). A necessary condition for the occurrence of an appreciable value of a nuclear factor in the forward direction is that the wave function of the excited state has a component of the same L-value and symmetry character as one of the components contained in the ground state. Otherwise the above-mentioned integration over all space coordinates will already give a vanishing result. This will always be the case for example if the excitation changes the parity, if configuration mixing is negligible and the wave function of the excited particle belongs to another configuration than in the ground state, or if the orientation of its orbital momentum with respect to the core has to be changed by the excitation.

If the space part of the wave function of the excited particle is not orthogonal to its ground state wave function, this may result in a more or less forward peaked angular distribution, depending on the amount of overlapping of both wave functions. Even if this overlapping is small, the cross section in the forward direction may be comparable with other single particle cross sections, because 100 % overlapping results in a cross section which may be an order of magnitude larger than the maximum in the case of orthogonal wave functions (compare fig. 1).

A simple example of such a situation is the elastic scattering. This may be considered in first order as the result of interfering contributions from all two-particle interactions in which the bound particle has been scattered back into its own level. The strong forward maximum in the elastic cross section of neutrons scattered on nuclei is understandable from this point of view.

The experimental differential cross sections in high energy nucleon-nucleon scattering show, except for the Coulomb scattering, a rather smooth dependence on the scattering angle. Assuming the same to be true for the matrix elements occurring in (9'), one may expect that the differential cross sections of the scattering of nucleons on nuclei will show an angular dependence similar to that of the nuclear factors. They will mostly be forward dipped except for cases in which the excitation may happen by pure spin or isobaric spin interactions.

As we remarked earlier, the Coulomb interaction, because of its strongly forward peaked matrix-element, might compensate the forward dip in the nuclear factor in the case of a dipole transition. The summed cross section for these excitations will be estimated in the next section, using the charge vibration model of ref. [12]). For cases in which the space part of the wave function of the excited state is not orthogonal to the ground state and therefore the nuclear factor still has an appreciable value in the forward

direction, the Coulomb cross section could become relatively large. Because the Coulomb interaction does not conserve the isobaric spin, it seems possible that such excitations could take place, although a closer investigation is necessary.

Summarizing our conclusions, we may say that mainly single particle levels are expected to be excited, and that their cross sections should fall off at large and generally also at small angles. More or less strong forward peaks could however occur if the cross sections are either caused by a) spin dependent interactions to states with spatial wave functions not orthogonal to the ground state functions and therefore with unchanged parity, or b) Coulomb excited dipole transitions with a change of parity and of one unit of orbital momentum.

Until now we have only been concerned with excitations appearing in the first Born approximation, in which only one particle changes its state in the independent particle model. In higher approximations the elastic scattering before and after such an excitation will become relatively important in directions in which the first order crosssection is small, but this should not change the general picture.

If an excited state has no component differing only in the state of one particle from the ground state, an excitation can still occur in second or higher order. A two-particle excitation, for example, may be caused by two types of matrix elements differing in their intermediate states. Firstly, the incident particle may have only one collision and push a nucleon into the continuum after which this nucleon scatters into a bound state by exciting a third nucleon. The angular dependence of this matrix element will be similar to the ones in single particle excitations and is expected to have a forward minimum. Secondly, the same state may be excited by a process in which the fast incoming particle undergoes two consecutive collisions. Because these two ways of excitation interfere, there is no special reason for a strong forward dip in the cross section, and the fall-off at large angles should be less steep than for a single particle excitation.

One weak point in the present treatment is the use of the Born approximation. It might be possible to improve the generality by applying the impulse approximation [2]), but the validity of this last method seems also doubtful because the final state of the struck nucleon falls inside the Fermi-sphere for the momentum distribution.

4. Collective Excitations

During the last few years it has turned out that some nuclear states may be understood in two ways, at first sight quite different. In the shell model they have wave functions which are mixtures of components in each of which only one particle is excited. In the collective model they are represen-

ted by certain collective excitations. Thus it has been possible to understand the frequent occurrence of a 2+ first excited state in even nuclei [8]) with both models [9,10]) and a similar situation is true for the "giant resonances" [11,12]

From such a point of view scattering with a collective excitation is not excluded in formula (9'). On the other hand it seems at present hopeless to get more than qualitative information from (9'), because of the questionability of the first Born approximation and the present lack of knowledge of the nucleon-nucleon interactions and of most wave functions.

Each of these factors is more favourable in the collective description of the vibrational states of spherical nuclei and the collective model is therefore more appropriate for calculating the cross sections for such excitations. For example in the special case of elastic scattering it is at least much simpler and more accurate to represent the interaction of the incoming particle by a smooth average nuclear field, than to consider the particle to be coherently scattered on all single nucleons in the nucleus.

Leaving out of consideration such questions as the applicability of the collective model to light nuclei [13]), we shall briefly consider two types of collective vibrations, namely the lowest quadrupole surface oscillation [13]) † and the dipole charge vibration as proposed to explain the "giant resonances" [12]). After having collected more experimental data, we hope to return to this matter.

The momentum change of a particle scattered by a potential well is taken up by the walls of the well. If these walls are able to oscillate, it is to be expected that such oscillations will be excited by scattering at not too small angles. Following ref. [13]), p. 15, we describe the state of the nucleus in a representation in which the operaters n_μ, representing the number of phonons having a component $\mu\hbar$ of angular momentum along the z-axis, are diagonal. The only other nuclear degrees of freedom considered are its centre of mass coordinates $\mathbf{R}_n$. For the first Born approximation we have to take the matrix element of the interaction of the nucleus with the incoming nucleon. Neglecting spin-orbit coupling and absorption, we take for this interaction

$$V(\alpha_\mu, \mathbf{r}-\mathbf{R}_n) = -V_0 \text{ for } |\mathbf{r}-\mathbf{R}_n| < R_0(1+\sum_\mu \alpha_\mu Y_{2\mu})$$
$$= 0 \text{ otherwise}$$

(the α_μ's are the coordinates of the nuclear surface as defined in ref. [13]), $Y_{2\mu}$ the normalized spherical harmonics, $\mathbf{r}$ the coordinate of the incoming proton and R_0 the radius of the spherical nucleus).

The computation of the space integrals in the matrix element and the subsequent summation over the coordinates of the nuclear surface result in the

† M. Goeppert-Mayer kindly mentioned during her visit in Sweden that she had performed some simple calculations concerning the angular distribution and polarization of the inelastically scattered protons exciting the first vibrational state in C^{12}.

following formula for the scattering cross section for the excitation of the lowest vibrational state, which will have its angular momentum orthogonal to $\mathbf{k}-\mathbf{k}'$:

$$\sigma(\mathbf{k}, \mathbf{k}') = \frac{k'}{k} \frac{5m^2 V_0^2 R_0^6}{2\pi B\omega\hbar^3} j_2^2(|\mathbf{k}-\mathbf{k}'| R_0), \tag{16}$$

where

$j_2(x)$ is the second spherical Bessel function,
$\hbar\mathbf{k}$ is the momentum of the incoming proton,
$\hbar\mathbf{k}'$ is the momentum of the scattered proton,
V_0 is the depth of the nuclear potential as felt by the incoming proton,
$\hbar\omega$ is the excitation energy,
B is a coefficient associated with the mass transported with the collective flow in the nucleus and is defined in ref. [13]).

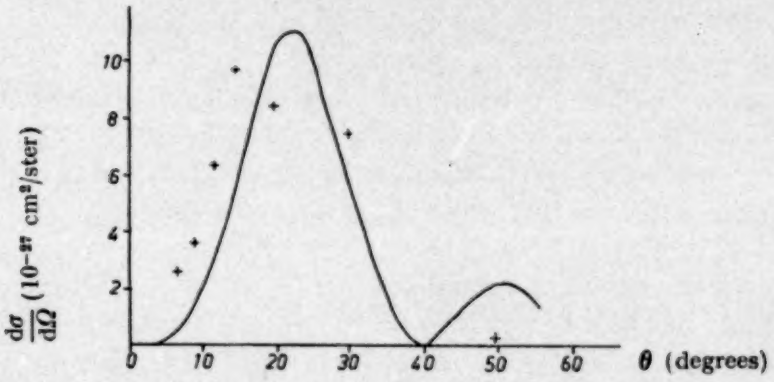

Fig. 3. The cross section of formula (16) in the text for the case of C^{12}. The constant B is taken 3 times its hydrodynamical value. Experimental values for the cross sections of the 4.43 MeV level of C^{12} are shown.

The cross section (16) for C^{12} is plotted in fig. 3. We have assumed 3 times the hydrodynamical value for B[13]), 14 MeV for V_0 [14]) and $12^{1/3} \times 1.2 \times 10^{-13}$ cm for R_0 [15]).

An accurate measurement of the polarization properties of the inelastically scattered protons could give a sensitive test of the applicability of the collective model to the first C^{12} level. We hope that such a measurement will be possible in this laboratory. If it turns out that the low lying 2+ states may be represented by quadrupole vibrations, the measured cross sections and excitation energies would allow a direct calculation of the quantities B and the deformability coefficients for several light nuclei.

The space integration in the first order matrix element also yields powers of the α_μ's higher than the first. These create higher vibrational states. Their magnitude is dependent on the assumed shape of the nuclear potential.

The second type of collective excitation we consider is the one proposed in ref. [12]) as an interpretation of the giant gamma resonances. If a proton

is scattered by the Coulomb potential of a nucleus, it reacts directly only on the nuclear charge and it is interesting to investigate how large the probability is that a charge vibration will be generated.

The nuclear protons are assumed to be homogeneously distributed over a sphere, which in the ground state coincides with the homogeneous neutron distribution. The centres of both spheres are assumed to be bound to each other by a harmonic force. For light even nuclei the first vibrational state of this system is assumed to have an excitation energy of about 20 MeV and has spin one and negative parity.

For a nucleus of Z protons and N neutrons, assuming the Coulomb potential to be that of a homogeneously charged sphere of radius $A^{1/3}\times 1.2\times 10^{-13}$ cm, a straightforward calculation in first Born approximation yields the cross section

$$\sigma(\mathbf{k},\mathbf{k}') = \frac{k'}{k}\cdot\frac{NZe^4}{N+Z}\cdot(\varepsilon\hbar\omega)^{-1}\cdot\left(\frac{3}{\varrho^3}\sin\varrho-\frac{3}{\varrho^2}\cos\varrho\right)^2\cdot\exp\left[-\frac{N}{Z(N+Z)}\cdot\frac{\varepsilon}{\hbar\omega}\right]. \tag{17}$$

The meaning of the symbols is

ε $= (\mathbf{k}-\mathbf{k}')^2\hbar^2/2m$,
$\hbar\omega$ — the excitation energy,
ϱ — $|\mathbf{k}-\mathbf{k}'|$ times the nuclear radius.

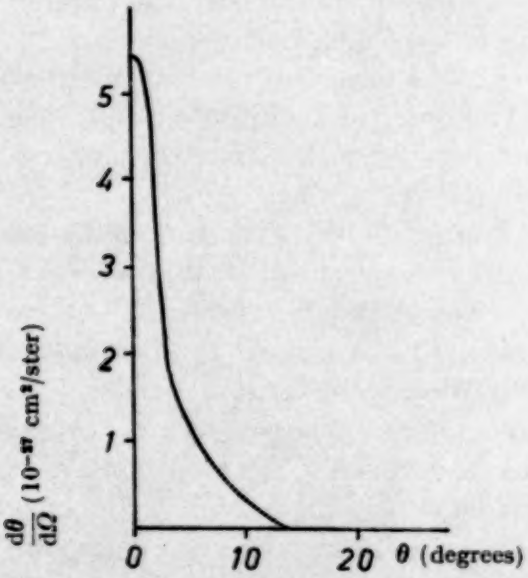

Fig. 4. The cross section of formula (17) in the text for the case of C^{12}.

Only the state with zero angular momentum component in the direction of $\mathbf{k}-\mathbf{k}'$ is excited. The cross section for C^{12} is plotted in fig. 4; the angular distribution has a sharp forward maximum. Because of the crudeness of

the model the result is only significant in so far as it shows that the possibility of a Coulomb excitation may not be neglected a priori and that its cross section should be forward peaked.

The "giant resonance" consists of a number of states which are probably single particle excitations [11]). In case such states also can be excited by nuclear interaction, the corresponding cross section will be extended to larger angles than is shown in fig. 4 for pure Coulomb excitation. If the matrix elements of both excitations are of the same order of magnitude an interference between the nuclear and Coulomb scattering may be expected. This will occur in the same angular region in which it is observed in the elastic scattering.

5. Comparison with the Measured C^{12} Spectrum

At this Institute preliminary measurements have been made on the inelastic scattering of 185 MeV protons on C^{12}. Levels of the following energies have been clearly resolved: 4.3 ± 0.2; 7.5 ± 0.2; 9.5 ± 0.2; 12.6 ± 0.2; 15.0 ± 0.2; 19.0 ± 0.3; 21.9 ± 0.3 MeV. Their differential cross sections are shown in fig. 3 of the preceding paper.

The first level is the well known 2+ level with an energy of 4.43 MeV [16]). In the independent particle model the orientation of the orbital angular momentum of one particle relative to that of the rest of the particles is turned by 180 degrees in comparison with the ground state. This means that the nuclear factor vanishes in the forward direction and a forward dip in the cross section is to be expected. At large angles all "single particle" cross sections should fall off. The measured angular distribution clearly confirms these expectations. The order of magnitude of the measured cross sections, however, cannot be compared with those resulting from (9') as long as the nuclear factors have not been calculated.

The angular distribution of the 4.43 MeV level also agrees quite well with that following from the assumption that a vibrational state is excited, as is seen from fig. 3. To get roughly the correct magnitude the constant B in (16) has to be taken 3 times its hydrodynamical value.

The second observed level is the 7.6 MeV 0+ level [16]). In the independent particle model it is considered as a two particle excitation and this would agree with its small and roughly isotropic cross section.

The third level has an energy of 9.6 MeV [16]). The angular dependence of its appreciable cross section shows the typical forward dip of most single particle excitations.

The 12.6 MeV level is probably identical with the known 12.76 MeV level [16]). Its cross section is too roughly determined to make a discussion possible.

The 15.0 MeV excitation is also observed at other bombarding energies

and is very probably the first $T = 1$ state of C^{12}, which is a $1+$ state [16]). Because of its parity it cannot be a dipole excitation and the forward peaked cross section could on the basis of our previous considerations be explained only by an overlapping of the space part of its wave function with the ground state function. In the case of pure L-S-coupling the ground state of C^{12} would be a ^{11}S-state. Because of the spin-orbit coupling this state is slightly mixed with a ^{13}P-state [17]). The first isobaric spin one level is generally assumed to be a ^{33}P-state [18]) with the same symmetry character as the ground state admixture [7]). Consequently there indeed seems to be an overlapping of the space parts of the ground state and 15 MeV excited state wave functions. The very sharp rise of the cross section in the forward direction suggests a Coulomb excitation. The cross section for this process could in principle be calculated from (9′), because in this case the rough wave functions and the interaction are known.

The 19.0 and 21.9 MeV levels seem also to be forward peaked. In the frame of the present treatment they could represent the other two states of the ^{33}P-triplet, but might also be dipole excitations. The broad peak in the energy spectrum which is observed at around 20 MeV (figure 2 of the preceding paper) shows indeed some resemblance to the peak of the "giant resonance" [5]). Such an interpretation seems not in disagreement with the cross section of fig. 4, taking into account the crudeness of the model used for the calculation and the possibility of excitation by nuclear interaction.

We are obliged to Professor T. Svedberg and Professor I. Waller for the interest they have shown in this work. It is a pleasure to thank Dr. W. J. Swiatecki and Dr. P. Hillman for stimulating discussions. The work has been supported by the Swedish Atomic Energy Commission.

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STUDIES OF INELASTIC SCATTERING OF 185 MeV PROTONS ON CARBON

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Abstract: Studies of the inelastic scattering of 185 MeV protons on carbon have been performed using a magnetic analyser. Differential cross sections of protons, exciting clearly resolved levels in C^{12}, are given.

1. Introduction

A magnetic analyser previously described [1]) was built in order to analyse high energy protons elastically and inelastically scattered on nuclei [2]), and secondary charged particles emitted in nuclear reactions.

Preliminary studies of inelastic scattering of 185 MeV protons on carbon have now been performed using this instrument. A brief description of the experiments and the results obtained will be presented. A more detailed description of the experimental set-up will be given in a later article.

2. Experimental Arrangement

The experimental arrangement is shown in fig. 1.

The proton beam of the Uppsala synchrocyclotron is extracted by the

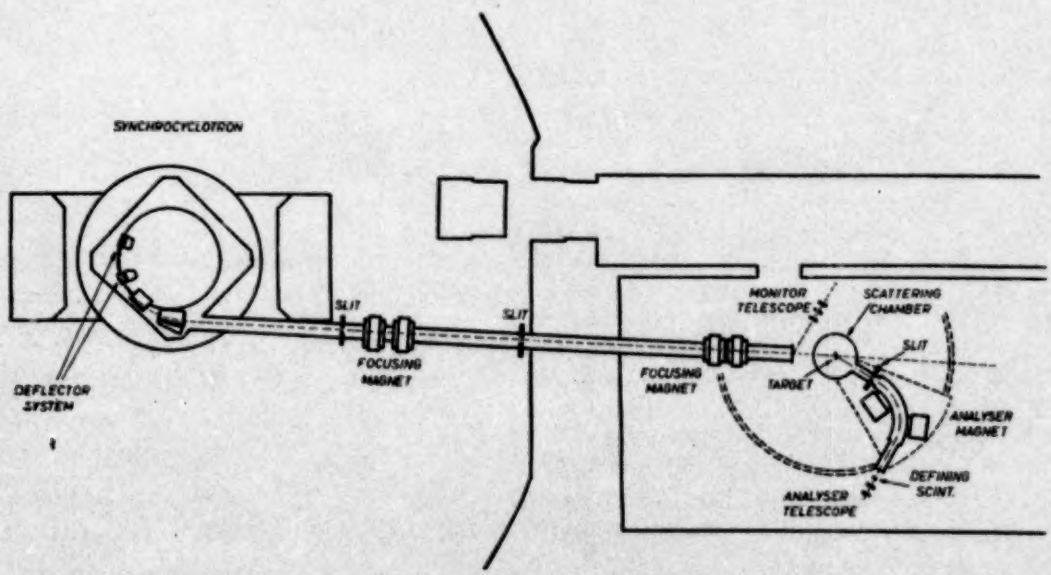


Fig. 1. Plan view of synchrocyclotron focusing magnets and magnetic analyser.

regenerator type of deflector suggested by Le Couteur [3]). The beam is focused by two quadrupole magnets to a round spot of approx. 5 mm diameter at the scattering center in the shielded experimental area. The rms. deviation from the central path at this point is not more than 0.5 degrees.

The incident protons have an energy of 185 MeV as determined by range measurements. The total width at half-height of the energy spectrum is believed to be not more than 1 MeV. The intensity of the focused beam is approximately 10^{10} protons per second. The repetition rate is 240 per second and the duration of a beam pulse is approximately 20 μsec.

The beam is monitored by a telescope of three scintillators which counts scattered protons from the aluminium window at the exit of the proton tube. The monitor is calibrated with a Faraday cup.

The target is positioned in an evacuated scattering chamber connected to the channel of the analyser magnet. The beam enters and leaves the scattering chamber through thin mylar windows. The scattered particles pass through a defining slit of 1° aperture at the entrance of the analyser magnet and are detected by a telescope of three scintillators in coincidence. The coincidence system has a resolving time of 10^{-8} sec. The first defining scintillator which is 3.5 mm wide is placed in the position of the image of the magnetic analyser. The paths of the protons of various energies through the analyser magnet have been studied with photographic films. The magnet characteristics are given in table 1.

TABEL 1

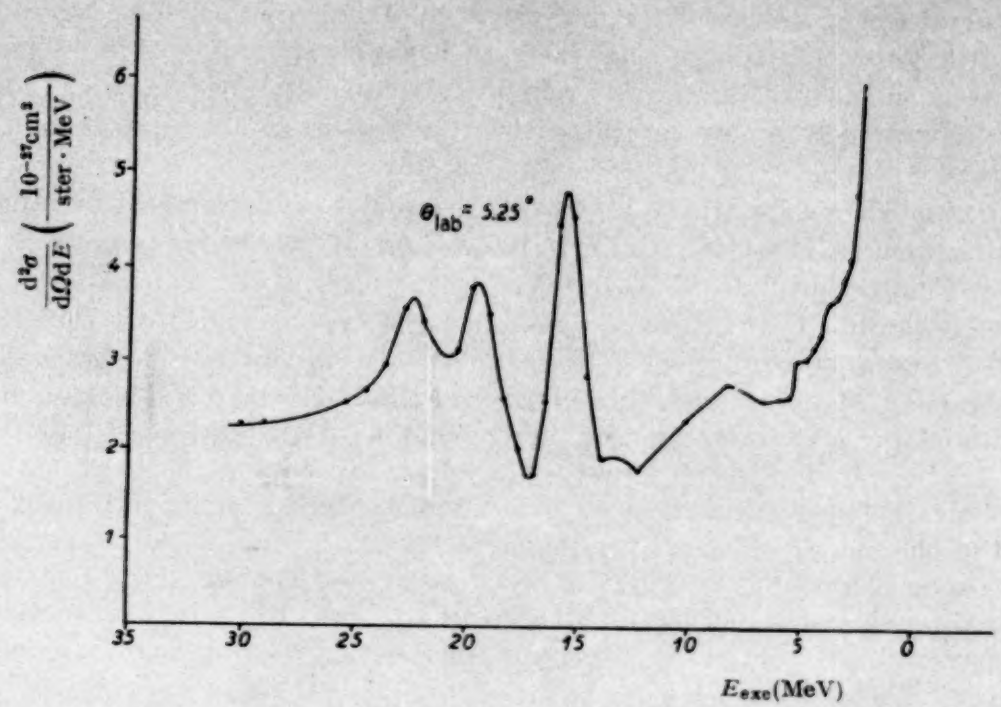
Analyser magnet characteristics

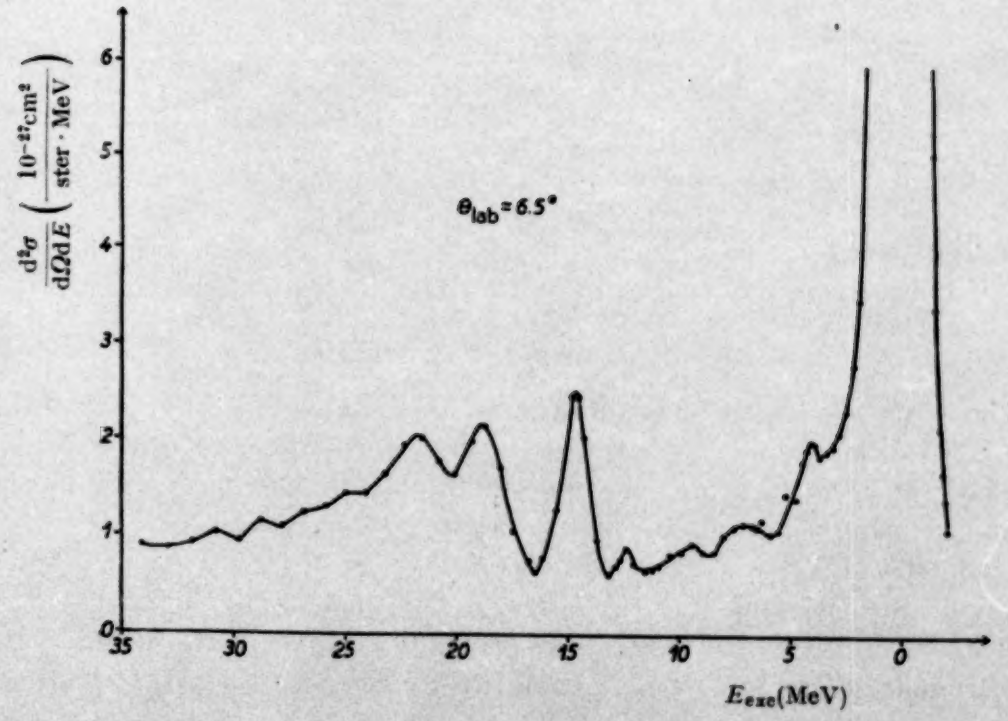
Radius of curvature	94 cm
Deflection angle	90°
Object distance	94 cm
Image distance	72 cm
Magnetic gap	16 mm
Entrance slit height	8 mm
Entrance slit width	15 mm

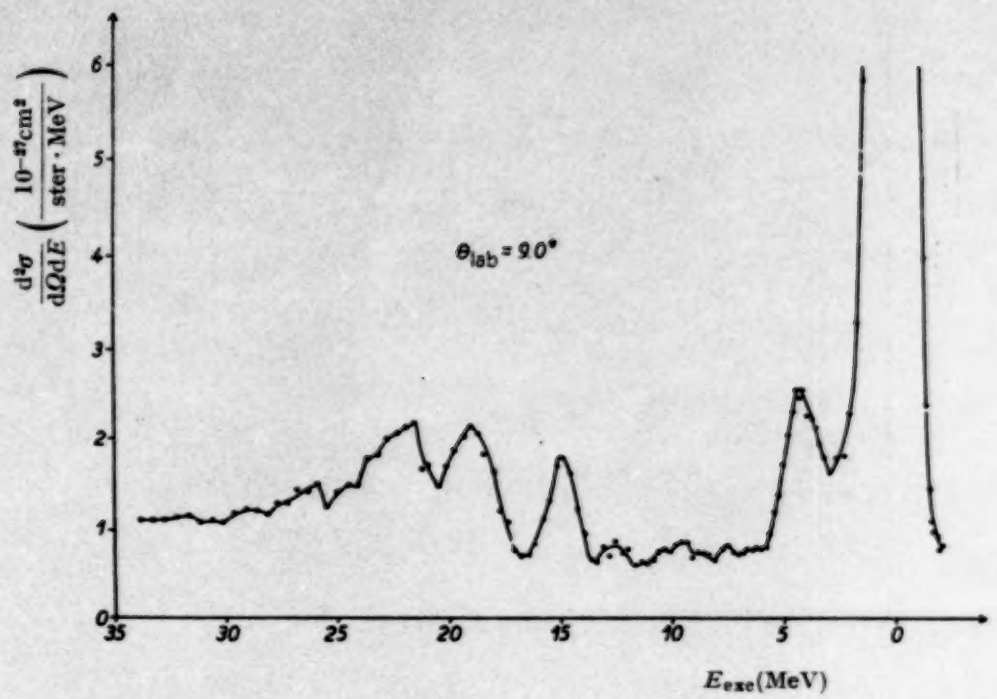
3. Experimental Procedure

The analyser magnet was calibrated with respect to energy by measuring the coil currents at which protons of known energies are observed. For these purposes we measured the peaks in the polyethylene spectrum resulting from free proton-proton collisions and the clearly recognized elastic, and 4.4 and 9.6 MeV inelastic peaks caused by protons scattered in C^{12}.

The transmission of the magnet was determined by measuring the areas of the peaks caused by free proton-proton scattering in the polyethylene spectrum at different angles. For these preliminary measurements a constant


$\frac{d^2\sigma}{d\Omega dE}\left(\frac{10^{-27}cm^2}{ster \cdot MeV}\right)$
$\theta_{lab} = 5.25°$
E_{exc}(MeV)


$\frac{d^2\sigma}{d\Omega dE}\left(\frac{10^{-27}cm^2}{ster \cdot MeV}\right)$
$\theta_{lab} = 6.5°$
E_{exc}(MeV)


$\frac{d^2\sigma}{d\Omega dE}\left(\frac{10^{-27}cm^2}{ster \cdot MeV}\right)$
$\theta_{lab} = 9.0°$
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35
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25
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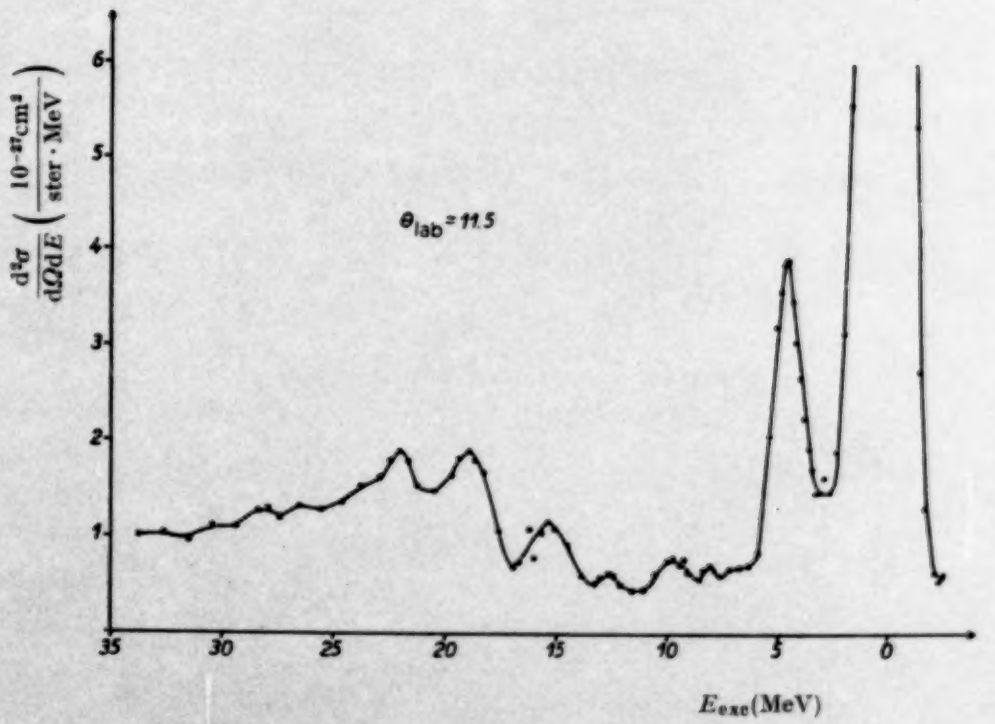
15
10
5
0
E_{exc}(MeV)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$\theta_{lab} = 11.5$
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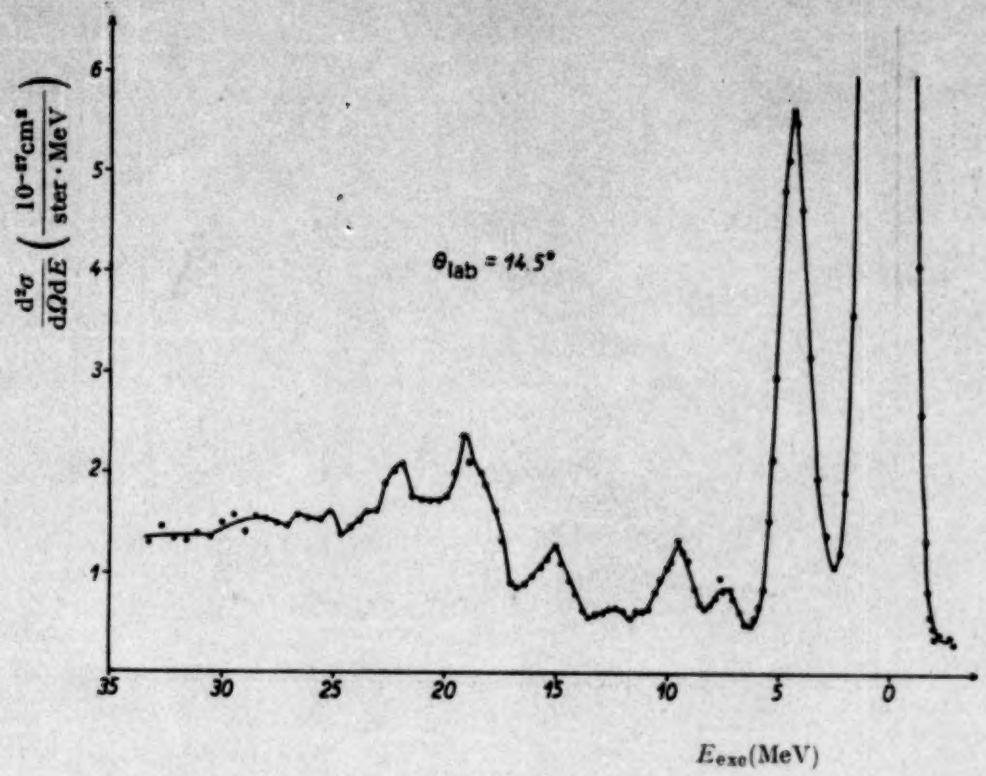
E_{exc}(MeV)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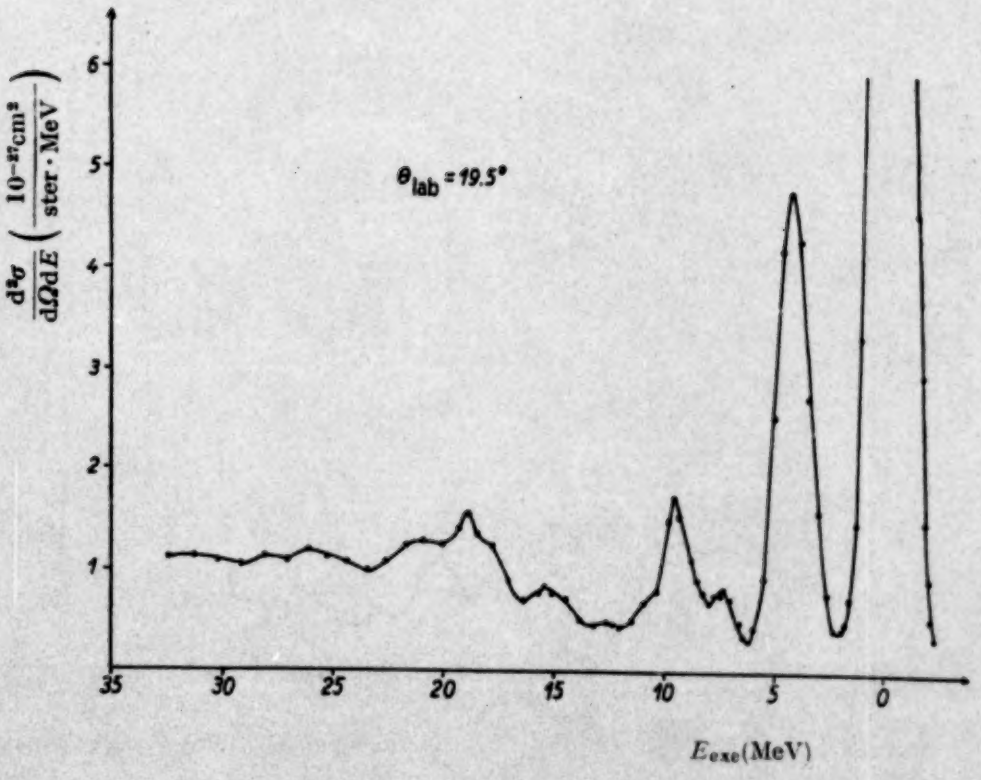
E_{exc}(MeV)


$\frac{d^2\sigma}{d\Omega dE}\left(\frac{10^{-27}cm^2}{ster \cdot MeV}\right)$
$\theta_{lab} = 19.5°$
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E_{exc}(MeV)

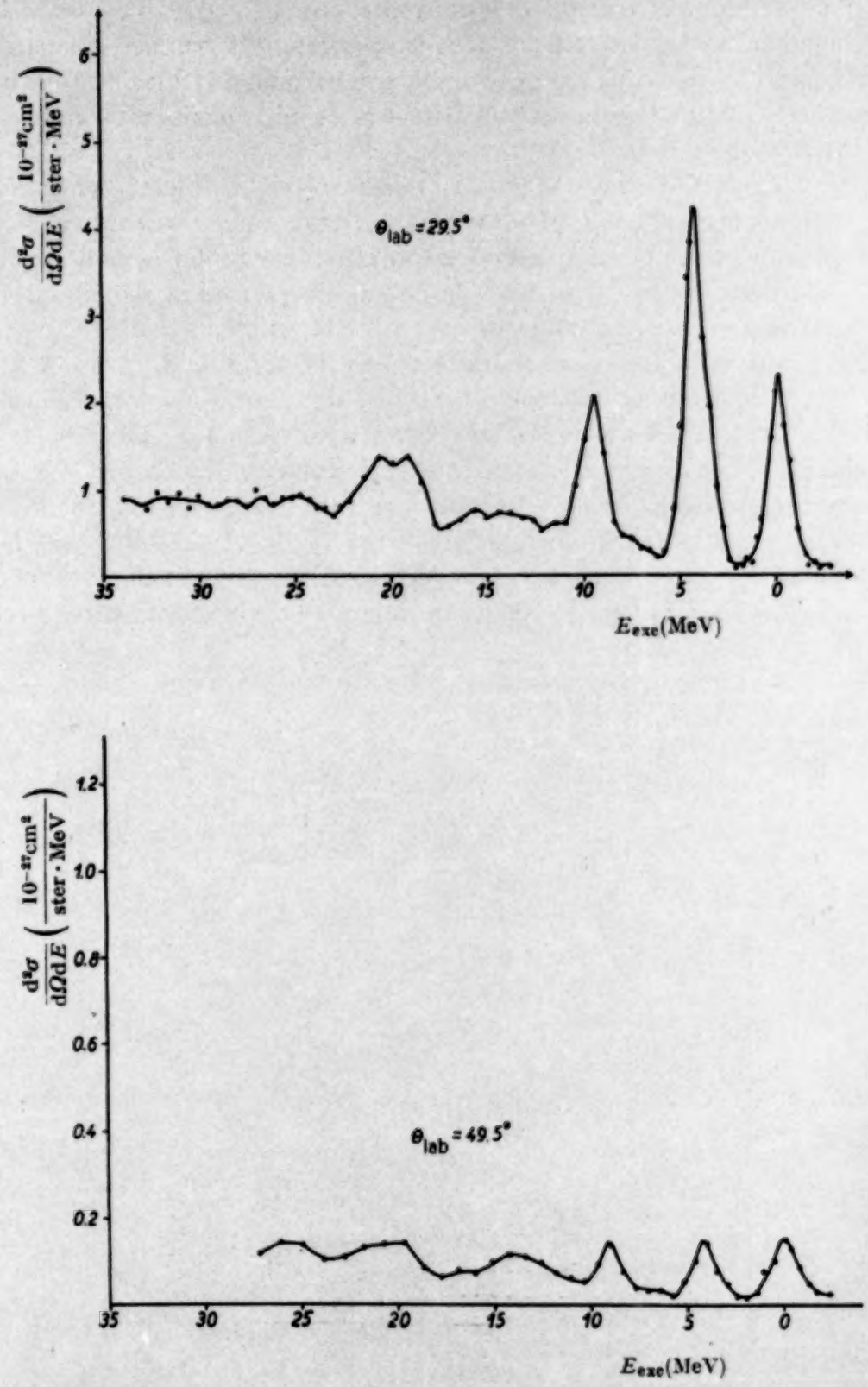

Fig. 2. Differential cross sections of 185 MeV protons inelastically scattered on C^{12} at different angles in the laboratory system. The abscissa indicates the excitation energy of the residual C^{12}-nucleus and the ordinate the cross section in the laboratory system.

(p-p) cross section of 3.4 mb/ster in the center of mass system was assumed †.

The actual measurements are done by counting the number of particles in the analyser telescope for a certain monitor reading. If hysteresis effects are avoided by running the magnet always in a similar manner, the measurements reproduce well.

Fig. 2 shows the energy spectra obtained for eight different settings of the analyser magnet angles. The average scattering energy is about 182 MeV. The energies of the plotted spectra are taken in the center of mass system. The cross sections and the angles, however, are plotted in the laboratory system, as these are the quantities of direct experimental interest. A plot in the center of mass system would not have been much different.

The transmission of the magnet decreased so suddenly for excitation energies larger than about 25 MeV, that a correction at these energies would have been very uncertain and was therefore not made. For this energy region the given cross sections are therefore much too small.

Levels of the following energies are found: 4.3 ± 0.2; 7.5 ± 0.2; 9.5 ± 0.2; 12.6 ± 0.2; 15.0 ± 0.2; 19.0 ± 0.3; 21.9 ± 0.3 MeV. The given errors in the energies are rms. errors. Fig. 3 shows the angular dependence of these levels.

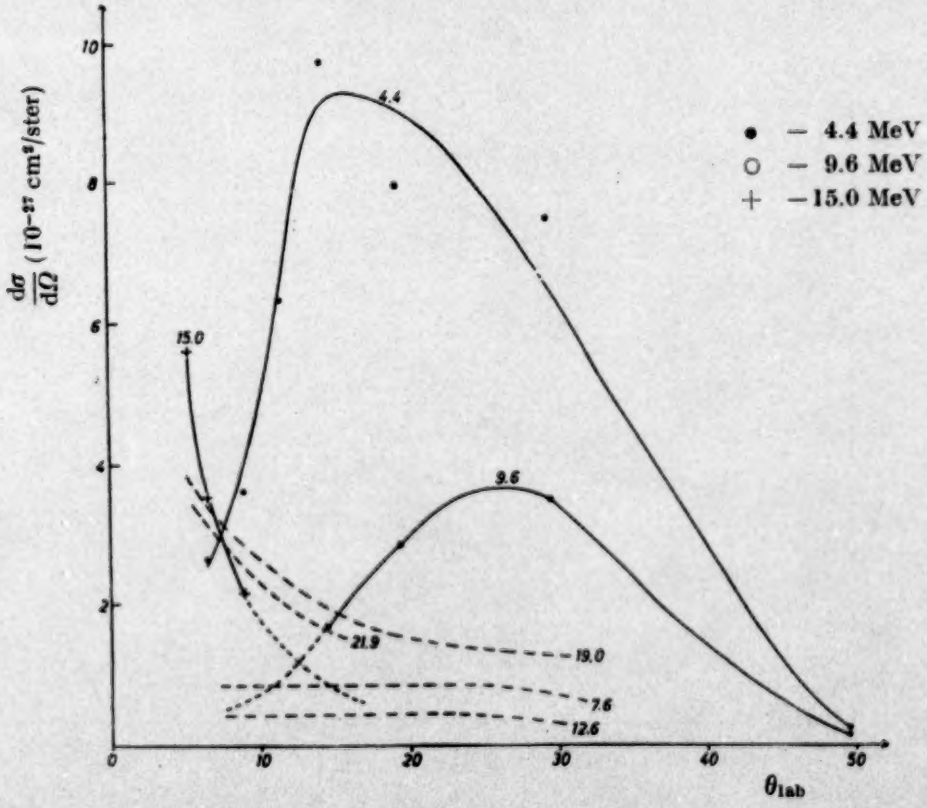

Fig. 3. The angular dependence of the cross sections of clearly resolved levels. The errors in the points are believed to be smaller than 20 %; the dotted curves may have errors of 75%.

† We thank Mr. Eric Taylor for sending us p-p cross section data.

We have not yet made a reliable error estimate. The cross sections in fig. 2 are believed to have errors smaller than 20 %. In fig. 3 the points belonging to the full curves are supposed to have the same maximum error; the errors in the dotted curves may be as high as 75 %, mainly because it is difficult to separate these peaks from the diffuse background consisting of unresolved levels and multiply scattered particles.

A further discussion of the results is given in the following paper.

We thank Prof. Svedberg for his interest and encouragement. The measurements were only made possible by the fine external proton beam, for which Ing. Å. Svanheden is responsible. We thank Mr. B. Smith for his able assistance and the cyclotron crew for their cooperation. The work has been supported by the Swedish Atomic Energy Commission.

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ON THE SPIN AND PARITY OF THE τ-MESON

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Abstract: Energy distribution curves of π^--mesons produced in τ^+-decays have been obtained by assuming that the isobaric spin of the three π-meson system equals unity and the ratio of probabilities of τ- and τ'-decays is 4. Comparison of the theoretical results with experimental data referring to 492 τ^+-decay events shows that the most probable values for τ-meson spin and parity is the 0^- combination.

As is well known, comparison of experimental data on the energy spectrum and angular distribution of π^--mesons produced in τ^+-meson decays with the theoretical curves computed by Dalitz [1]), leads to the conclusion that the most probable value for the τ-mesons spin and parity is 0^-. This leads to the so-called "$\tau-\theta$-problem" whose essence is that $\tau^\pm$-mesons and $\theta^\pm$-particles (which decay into two π-mesons) cannot be identical particles despite the fact that their masses and lifetimes are equal within the experimental errors (possible spin and parity values of the θ-particles are 0^+, 1^-, 2^+ etc.; see ref. [2])).

Dalitz's curves were obtained on the assumption that π-mesons from τ-meson decay are formed in states with the lowest possible orbital momenta. The correctness of this assumption is not obvious and Marshak [3]) has indicated in this connection that agreement between the experimental and theoretical π-meson spectra can be obtained if the τ-mesons spin and parity are assumed to equal 2^+ and the π-mesons to be emitted with orbital momenta (2,1) or (2,3) (the first figure in the parenthesis indicates the relative orbital momentum of two π^+-mesons and the second the π^--meson orbital momentum relative to the centre of mass of the π^+-mesons). In this case the aforementioned "$\tau-\theta$ problem" does not arise.

In the present paper it will be shown that the experimental data exclude the possibility examined by Marshak if some additional considerations (which hitherto have not been taken into account) are invoked. Moreover, it is found that the theoretical curves to be compared with the experimental data agree with those computed by Dalitz only for 0^- and 2^-. For other parity and spin values (1^+, 1^-, 2^+) curves must be used which appreciably

differ from Dalitz's and even more from those plotted on basis of the experimental data.

The considerations from which we proceed consist of the following:

1) It is assumed that the isobaric spin $I_{3\pi}$ of a system of 3 π-mesons created in a τ-meson decay is equal to 1. This assumption which has already been made by some other authors (Gell-Mann [4]), Wentzel [5])) directly follows from Gell-Mann's scheme [6]), according to which a τ-meson has an isobaric spin $I_\tau = \frac{1}{2}$, and the slowness of the reaction

$$\tau^+ \to \pi^+ + \pi^+ + \pi^- \tag{1}$$

is explained by non conservation of isobaric spin in this process. It is natural to assume that the smallest possible change in isobaric spin occurs in this case,

$$\Delta I = \tfrac{1}{2}, \tag{2}$$

and hence $I_{3\pi} = 1$.

2) It is assumed that K-mesons, undergoing so-called τ'-decay,

$$\tau'^{\pm} \to \pi^{\pm} + 2\pi^0, \tag{3}$$

and τ-mesons are identical particles.

3) According to experimental data [7, 8, 17])† the ratio of the probabilities of τ- and τ'-decays is approximately 4:

$$W_\tau / W_{\tau'} \approx 4. \tag{4}$$

Dalitz and Berestetsky [10]) have shown independently that if assumptions 1) and 2) are correct the following formula holds for the ratio $W_\tau / W_{\tau'}$:

$$W_\tau / W_{\tau'} = (4F + \phi)/(F + \phi) \tag{5}$$

where F is a quantity which is obtained on integrating over π-meson energy the squares of the moduli of matrix elements symmetrical in the momenta of all three π-mesons; ϕ is an analogous quantity which is obtained with matrix elements symmetrical only in the momenta of identical π-mesons. Comparing (4) and (5) we arrive at the conclusion that

$$\phi \approx 0 \tag{6}$$

or, in other words, the π-mesons are created in states which are symmetrical with respect to the momenta of all three particles. It should be noted that

† The experimental data are not very accurate. The error is not indicated in ref. [7, 8]) and in ref. [17]) the result is $W_{\tau'}/W_\tau = 0.25 \pm 0.12$. Analysis of stability of the results obtained in the present paper with respect to small deviations from (4) is difficult because of the presence of an interference term resulting from superposition of states symmetrical and antisymmetrical with respect to the π^+ and π^- meson variables, for which the phase shift is unknown. It should be mentioned that the difference between the phase volumes for τ and τ' decays which is the result of a small difference in the $\pi^\pm$ and π^0 meson masses has not been taken into account in equation (5). However, due to the low precision of the experimental data this discrepancy is not significant.

if this is indeed the case the spectrum of π^+-mesons produced in τ'-decays should be identical with that of π^--mesons produced in τ-decays.† The lowest orbital momenta corresponding to these states are presented in table 1 in which, for the sake of comparison, we also give the orbital momenta and types of matrix elements employed by Dalitz. As can be seen from the

TABLE I

Spin and parity of τ-meson	Minimal (l, l') pairs for symmetrical states	Minimal (l, l') pairs and types of matrix elements employed by Dalitz [1])
0^-	(0,0) F	(0,0) F
1^+	(2,1) F	(0,1) ϕ
1^-	(4,4) F	(2,2) ϕ
2^+	(2,3) F	(2,1) ϕ
2^-	(0,2) (2,0) F	(0,2) (2,0) F

table, the only limitation imposed by Dalitz in choosing the various types of matrix elements was that the magnitude of the orbital momenta should be minimal; only in two cases—0^- and 2^-—the matrix elements correspond to the symmetrical states mentioned above.

In fig. 1 are shown π^--meson spectra computed exclusively on basis of matrix elements for symmetrical states possessing the orbital momenta indicated in the second column of table I (concerning the formulae employed in the calculations see Mikhailov's paper [11])). A histogram of the experimental data referring to 492 cases of τ^+-mesons decay, ref. [12–21]), is shown in the same figure.†† As can be seen from the figure, the curves corresponding to spins and parities 1^+, 1^-, 2^+ are totally different from the experimental spectrum. The shape of the curve corresponding to 2^- is

TABLE II

τ spin and parity	χ^2	P_{χ^2}
0^-	7.53	0.188
2^-	25.92	$\approx 10^{-4}$
1^-	68.49	$< 2\times10^{-16}$
1^+	112.32	$< 10^{-16}$
2^+	135.55	$< 10^{-16}$

† At present 34 events of τ'-decay have been observed in which the π^+-meson energy could be ascertained. These data, however, are insufficient for comparison with the theoretical curves because of the low statistical precision.

†† After the present investigation was completed a paper [22]) appeared in which 481 τ-decay events were examined. However, these experimental data are compared with the theoretical curves obtained without taking account of conditions 1)–3) considered in the present work.

more similar to that of the experimental distribution but is similar to it to a less degree than the curve for a 0^- spin and parity. These qualitative conclusions have been confirmed by calculations of χ^2 and the Pearson probability P_{χ^2} given in table 2.

Summarizing, we draw the following conclusions:

1) The most probable values for the τ-meson spin and parity is the 0^- combination.

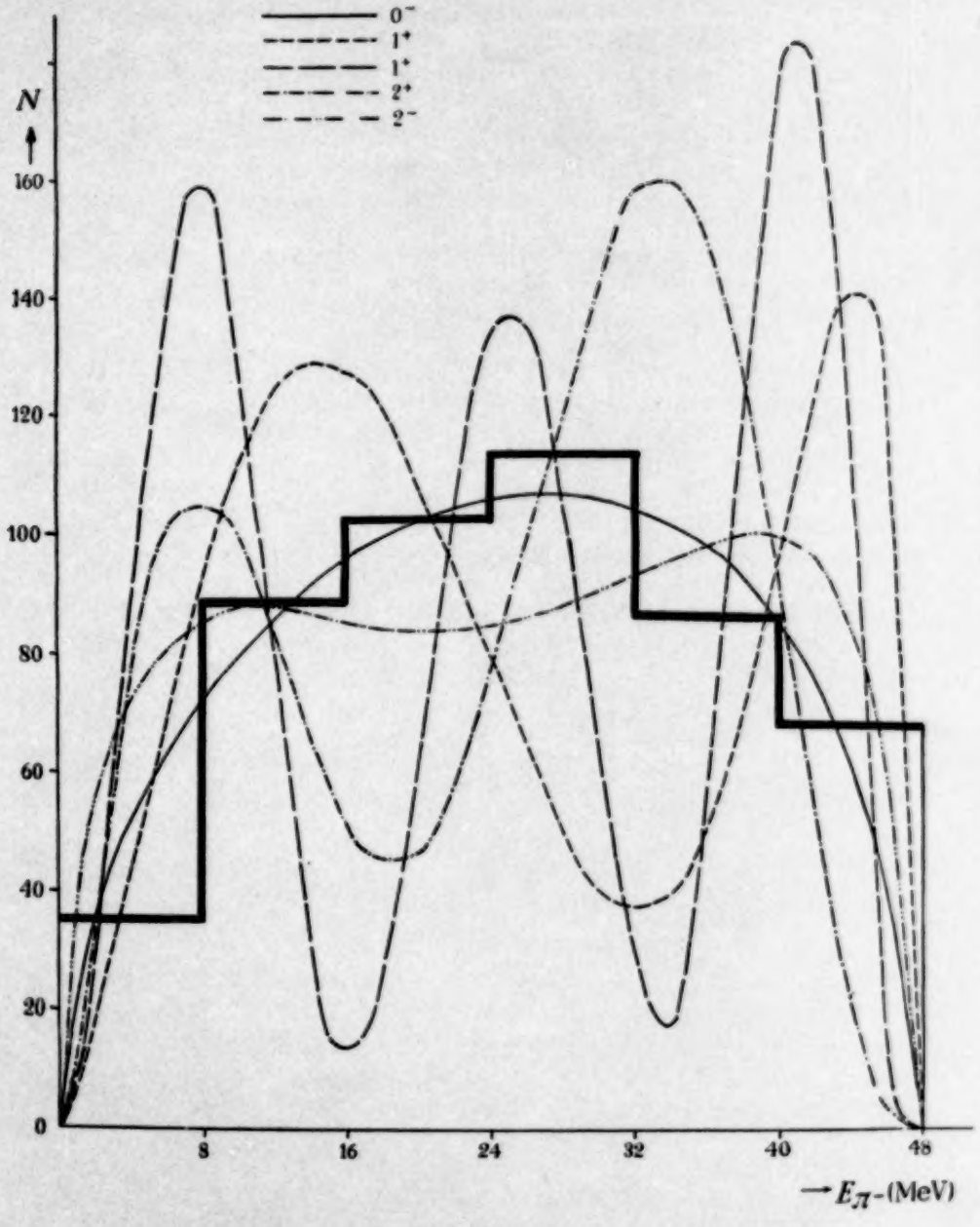

Fig. 1. Theoretical curves and experimental histogram for energy distribution of π^--mesons from τ^+ disintegration for various τ-meson spin and parity values; the curves and histogram are normalized for 492 decay events.

2) The combinations 1^+, 1^-, 2^+ should be considered to be practically excluded.

Thus the most probable τ-meson spin and parity values are just those which lead to the "$\tau-\theta$ problem".

In conclusion we express our sincere appreciation to Professor I. I. Gurevich for an interesting discussion of our results.

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FURTHER ANALYSIS OF THE NEUTRON COMPONENT OF THE BACKGROUND OF COUNTERS USED FOR ^{14}C AGE MEASUREMENTS

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Abstract: The background of counters, used for ^{14}C age measurements, has been reduced appreciably by a special arrangement of blocks of paraffin wax (mixed with boric acid) inside the iron shied. The background could be divided into a constant fraction, probably due to radioactive contamination of the counter, and a fraction, depending on the barometric pressure, due to locally produced neutrons. The neutron background has been reduced by a factor of about 7; the barometer effect was 10 per cent of the neutron component per cm Hg in all arrangements studied. The relative increase of the neutron component in the ^{14}C counter during the solar flare of February 23 has been the same as in the neutron monitor.

If a counter is carefully shielded by a thickness of iron or lead to absorb gamma radiation and by an anticoincidence ring to suppress the effect of mesons, a residual background count remains. In a previous article [1]) evidence was given that neutrons, produced by cosmic radiation in the iron shield. contributed appreciably to this residual background. Recent improvements of the shielding of the counters have reduced this neutron effect; moreover it is possible now to give a more reliable analysis of the residual background. In the following discussion we will confine ourselves mainly to our largest counter (length 51.5 cm, diameter 7.0 cm, recent carbon count at 3 atm: 37.0/min). It is of importance to mention here that the present apparatus is located at ground level in a one floor building. So the building itself does not contribute to the shielding.

Since the neutrons are mainly produced in the iron of the shield itself, a layer of paraffin wax, mixed with boric acid (15 per cent by weight), on top of the shield is not very effective (see ref. 1) and also table 1, column 1—2). Therefore the paraffin wax (plus boric acid) was placed now inside the shield. Fig. 1 shows the present arrangement. The vertical blocks came in place of iron blocks (thickness 10 cm). The thickness of the horizontal layer is 15 cm. Though it might have been of importance to have also paraffin wax under the counters, this was not yet done for technical reasons.

The backgrounds, measured in the various arrangements, have been summarised in table 1. The barometer effect will be considered first. It

decreased from 0.37 to 0.05 (column 1—7) whereas the decrease of the background is from 5.50 to 2.44 only. This suggests that only part of the

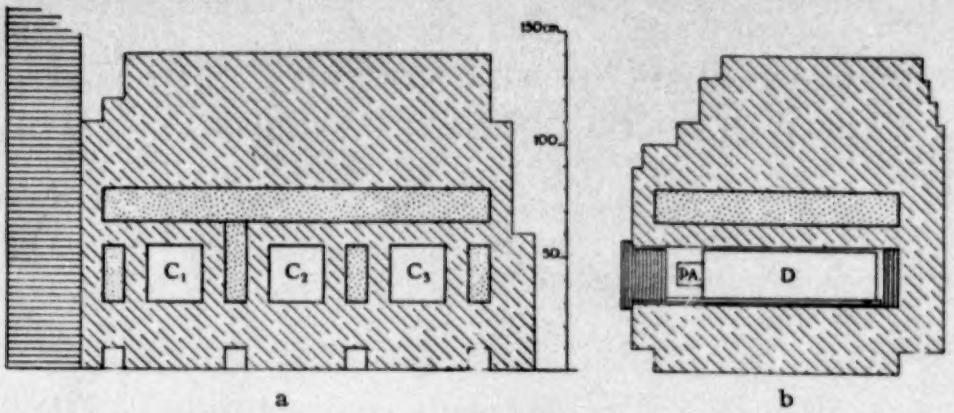

Fig. 1a. Front view of the shield. C_1-C^3: holes for the counters. /////// iron. ▒▒▒ paraffin wax, mixed with borid acid. ═══ brick wall. ||||||||| lead. Fig. 1b. Section through C_2.

background is due to neutrons. According to various data published, the barometer effect of the locally produced neutrons is about 10 % per cm change of barometric pressure. In our previous report [1]) a lower barometer effect of only 6 % per cm Hg was assumed to occur under the layer of 40 cm of iron of the shielding, since generally the hardness tends to increase with increasing thickness of the absorber. Recently, however, we checked that the intensity under 2.5 metres of sand still agreed very well with the absorption coefficient measured by Reich [2]) for thinner layers (see also ref. 4)). Assuming now a barometer effect of 10 % per cm Hg for the neutron background in our counter, the data in table 1 can easily be related with each other. The barometer effect of 0.05 in column (7) suggests that only

TABLE I

Counting rate, in counts per minute, for the large counter, in hole C_2 (fig. 1) under various conditions at a barometric pressure of 766 mm.

	(1) 40 cm Fe no P	(2) 40 cm Fe 20 cm P	(3) 40 cm Fe 20 cm P 15 cm Pb	(4) 40 cm Fe 20 cm P no cyl.	(5) 40 cm Fe P inside no cyl.	(6) 50 cm Fe P inside no cyl.	(7) 70 cm Fe P inside no cyl.
CO_2 (3 atm)	5.50 (3.70) 0.37±0.05	5.10 (3.30) —	4.55 (2.75) —	4.25 (2.45) 0.25±0.04	3.00 (1.20) —	— — —	2.44 (0.64) 0.05±0.015
propane (2 atm)	12.5 (10.7) 0.8±0.3	10.9 (9.1) 1.0±0.1	9.90 (8.10) —	— — —	5.30 (3.50) —	4.47 (2.67) —	4.00 (2.20) —

Between brackets: background after subtraction of the (constant) contamination effect (see text). The barometer effect (third line) is given in counts per minute per cm change of barometric pressure. P = paraffin wax, mixed with boric acid. No cyl. = no paraffing wax around the counter for electrical insulation.

about 0.5/min is due to neutrons. Consequently about 1.8/min is due to other sources, probably radioactive contamination of the counter or its shielding. Subtracting tentatively this constant effect of 1.8/min from all counting rates, the figures between brackets represent the background due to neutrons. It is interesting to note that the barometer effect is now 10 % per cm Hg, within the limits of error, for all arrangements (column 1, 2, 4 and 7). For the same counter, filled with propane, the errors were larger since these measurements were not extended over periods with appreciable variations in barometric pressure.

The reduction of the residual background with increasing thickness of iron above the counter is smaller than the reduction calculated from the coefficient of absorption in iron (see Reich [2]) — at least if the total background is ascribed to neutrons. For the fraction ascribed to neutrons according to the new analysis, however, the absorption of the primary cosmic radiation in the iron shield agrees very well with the data published by Reich (l.c.).

The effect of the displacement of the paraffin wax to the inside of the shield is somewhat larger for the CO_2 than for the propane filling (column 1—5). This is due to the fact that propane detects the neutrons mainly by recoils of *fast* neutrons whereas a counter filled with CO_2 detects mainly gamma rays produced by capture of *slow* neutrons. It is obvious that the effect of paraffin wax (plus boric acid) need not be the same for these two groups of neutrons, though it is difficult to estimate this difference quantitatively.

The improvement of the shield has also reduced the background of the other two counters. The background of the quartz counter (length 37.0 cm, diameter 5.0 cm, net recent carbon count 14.6/min) was reduced from 1.6/min in the arrangement of column (2) to 0.92/min (column 7). This counter is still insulated by a cylinder of paraffin wax, thickness 2 cm. Since this counter is not in use for very old samples it is not of interest to reduce the background further by removing this cylinder. The present barometer effect is difficult to measure; it is about 0.015 ± 0.015/min per cm Hg.

For the large counter a further reduction of the background would be of more interest. It was checked recently that the residual background of 1.8/min was not due to a horizontal component of the cosmic radiation, escaping from the anti-coincidence ring. A set of counters in front of the ends of the anti-coincidence ring gave no improvement (0.00 ± 0.03/min).

The effect of the solar flare [3]) of February 23 of this year is also better understood by the present analysis of the background. Between 3.45 and 7.20 G.M.T. the average counting rate in our neutron monitor was nine times normal, whereas the large counter considered here gave an average

background of 19 ± 2/min instead of 4.25/min (column 4, table I). Previously the total background was ascribed to neutrons. This implied the increase of the neutron effect in he CO_2 counter to be $19/4.25 = 4.5$; this is only one half of the increase in the neutron monitor. Ascribing, however, 1.8/min to contamination, the neutron effect in the CO_2 counter would have increased by a factor of 17/2.45 .Though it is still somewhat less than a factor of nine, the difference is no longer very significant. So there is no reason to assume the solar effect detected by the neutron monitor to be essentially different from the effect detected under a layer of 40 cm of iron.

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PHOSPHORUS ISOTOPES P^{32} AND P^{33} IN RAIN WATER

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Abstract: In 1955, Marquez and Costa [1]) detected Phosphorus 32 (half life ≈ 14 days) in rain water. They attributed its presence to the spallation of atmospheric Argon nuclei by cosmic rays. We have detected, in rain water, the second short lived radio-isotope of Phosphorus, P^{33} (half life ≈ 25 days).

Approximate concentrations (atoms ml^{-1} of rain water) of the two isotopes have been obtained; both are present in comparable amounts. The observed concentrations agree in order of magnitude with those reported by Marquez and Costa for P^{32}. They agree also with a rough estimate based on the number of disintegrations produced in the atmosphere by cosmic radiation and the estimated cross section for this reaction.

A simultaneous determination of the concentrations in rain water of these chemically identical radioactive nucleides, may prove valuable in meteorological studies. It may be useful for tracing the large scale motion of air masses.

1. Introduction

The interactions of the primary and secondary particles of the cosmic radiation with the nuclei of the atmosphere lead to a continuous production of a variety of stable and unstable forms of matter. The production rates are small and therefore it has not yet been possible to detect stable end products of these reactions (with the possible exception of He^3).

The unstable nuclei can, however, be detected owing to their radioactivity and in recent times have claimed the attention of several investigators. The best known of the cosmic ray produced isotopes are carbon 14 [2]) and tritium [3]). C^{14} has been extensively used to date organic relics of geological or archeological interest and tritium analysis has found application in problems of hydrology.

In 1955, Peters [4]) calculated the probable abundance of cosmic ray produced Be^{10} (half life 2.7 million years) in the atmosphere and in rain water and predicted that the resulting radioactivity may be strong enough to be detected in slowly accumulating sediments on land and on the ocean floor. Be^{10} analyses may be useful for dating geological processes of the late tertiary. Evidence has recently been obtained for the presence of Be^{10} in Pacific Ocean sediments [5, 6]). The abundance calculations [4]) apply equally well to the short lived isotope Be^7 (half life 53 days) which has recently been detected in rain water [7, 8]) and air borne dust [9]).

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All these isotopes, C^{14}, Be^{10}, Be^{7} and T, are produced in cosmic ray interactions with the main constituent nuclei of the atmosphere, namely, nitrogen and oxygen. Progress in this field of cosmic ray produced activities was made by Marquez and Costa [1]), who showed that in spite of the low abundance of Argon in the atmosphere ($\approx 1\,\%$), the production of the radio-isotope P^{32} resulting from the spallation of Argon nuclei by cosmic rays is nevertheless sufficient for its detection in rain water.

We have now detected the second isotope of Phosphorus, P^{33}. This isotope is more difficult to detect because in addition to weak concentration, one deals with a very soft β-radiation compared to that of P^{32}. In this paper we present evidence for the existence of the isotope P^{33} in rain water. The values obtained for the absolute concentrations are still fairly rough.

Because of their identical chemical behaviour and favourable half-life periods, the simultaneous determination of P^{32} and P^{33} concentrations should prove valuable for tracing large scale motions of air masses, since they are produced at a constant rate which is obviously independent of meteorological conditions and is a function of atmospheric height and of latitude only.

2. Experimental Procedure

The Phosphorus isotopes were extracted from rain water samples of 50 litres each. They were obtained in the form $Mg_2P_2O_7$ by a chemical procedure identical to that described by Marquez and Costa [1]).

A pilot experiment showed the presence of some stable Phosphorus (1—2 mg of Phosphorus in 50 litres of rain water). Most of this might have been contributed by the chlorinated rubber paint on the surface on which the rain water was collected. In later experiments this contribution was eliminated by covering the collecting surfaces with polyethylene sheets.

The energetic β-radiation from P^{32} ($\beta_{max} = 1.7$ MeV) makes its detection simple, but in the case of P^{33} the β-radiation is much softer ($\beta_{max} = 0.25$ MeV) and self-absorption in the source creates some difficulty. In order to obtain sources of high specific activity for the efficient counting of P^{33} we did not add any stable Phosphorus as carrier over and above the stable Phosphorus already present. Three samples were analysed and contained residues of 3.6, 8.0 and 2.8 mg of $Mg_2P_2O_7$ respectively. The chemical yields are still uncertain, but should lie between 50 and 80 %.

The final purified samples were deposited on perspex sample holders and counted on a low level β-counting set-up which employs an end window counter as detector.

The rain collection dates and details of the sources and counters are summarized in table 1.

TABLE 1

Sample No	S.1	S.2	S.3
Date of rainfall	4.7.1956	12.7.1956	26.7.1956
Vol. of water collected in litres	50	50	50
Wt. of $Mg_2P_2O_7$ residue (mg)	3.6	8.0	2.8
Fraction deposited for counting	23.6 %	20 %	100 %
Counter used	A	A	B
Thickness of film covering the sample (mg/cm²)	1.1	0.6	1.4
Window thickness (mg/cm²)	1.6	1.6	2.4
Window diameter (cm)	0.7	0.7	2.5

Counters A and B gave consistent backgrounds of 7.5 ± 0.4 and 62.0 ± 2.0 counts per hour respectively. Counter efficiencies were determined by counting a Tl^{204} source of known strength under conditions identical with those employed for the rain water samples. The efficiencies of counters A and B were found to be 13.5 ± 1.0 and 23.2 ± 1.0 percent respectively.

The rain water samples were counted with various absorber thicknesses and recounted after a period of fifteen days.

For comparison, absorption measurements on artificial P^{32} were made under identical conditions of source mounting and geometry.

Since each set of absorption measurements takes a few days, the counting rates had to be corrected for the decay during the period of measurements. The net counting rates as a function of absorber thickness for the rain water samples and artificial P^{32} activity are shown in figures 1 and 2.

3. Interpretation of the Results

We have studied the absorption curve for the sample S.3 in greater detail than the others and shall therefore discuss it more fully.

In fig. 1, we have plotted on a semi-log scale the net counting rates of sample S.3 below various absorber thicknesses on the 8th and the 22nd of August, 1956. The initial sharp drop and the subsequent slow decrease in the counting rates with increasing absorber thickness shows the presence of a low and a high energy β-group in the rain water sample.

For comparison, we have plotted the absorption curve of an artificial source of P^{32}. The spectrum is within statistical errors identical to that observed for the rain water activity for absorber thicknesses greater than 28.2 mg cm^{-2} of Al (fig. 1).

We now consider separately the part of the curve obtained with absorber

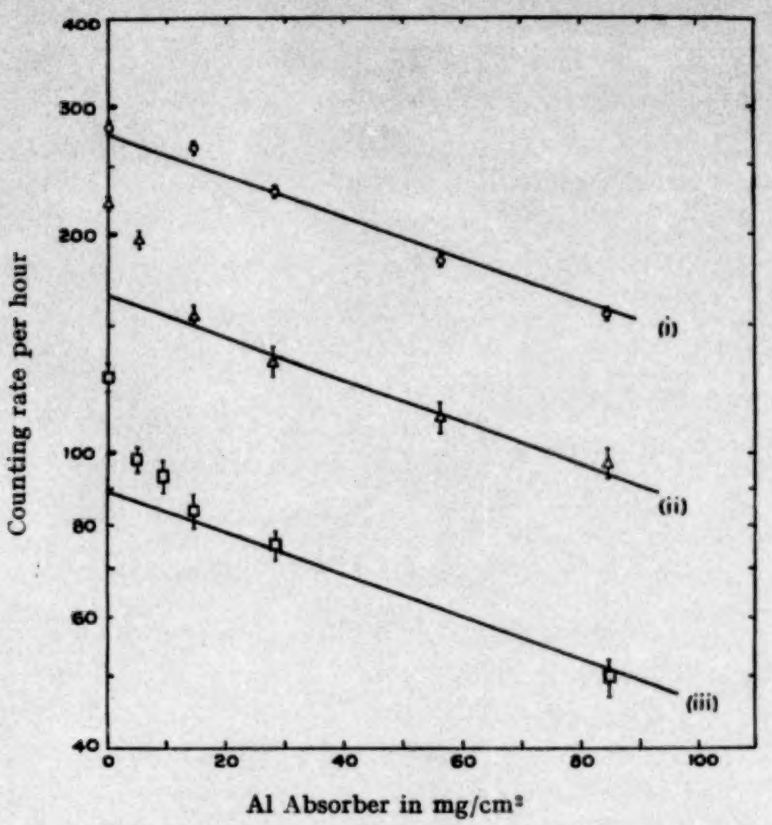

Fig. 1. Counts per hour are plotted against absorber thickness in mg/cm² of Aluminium
(i): absorption curve for a sample of artificial P^{32}.
(ii) and (iii): absorption curves of rain water sample S.3 taken on two different dates: 8th August, 1956 and 22nd August, 1956.

thicknesses less than 28.2 mg cm^{-2}, which contains some low energy β-component. We subtract the hard component by extrapolating the activities for absorber thicknesses larger than 28.2 mg cm^{-2} back to zero thickness. The remaining activity at small absorber thicknesses, if interpreted as due to a single low energy β-component, has an absorption coefficient of 138 ± 24 cm^2 g^{-1} of Al. This is consistent with the value of 123 cm^2 g^{-1} of Al for P^{33} as calculated from a formula given by Gleason *et al.* [10]).

Comparing the counting rates on the 8th and 22nd of August, 1956, we obtain a half life of 23.4 ± 6.7 and 15.6 ± 1.1 days for the low and high energy components respectively; both of these are consistent with the values reported for P^{33} (half life 25 days [11]), and P^{32} (half life 14.3 days [12])).

The sample S.3 was subjected to one additional cycle of chemical purification. The sample was dissolved in 1:1 HCl and the Phosphorus was precipitated as Magnesium ammonium phosphate. This precipitate was then washed with dilute ammonia and filtered through a Whatman's No: 42 filter paper. The filter paper along with the precipitate was ignited to obtain $Mg_2P_2O_7$. This was dissolved in 1 : 1 HCl, deposited and counted. The yield was 70 ± 22 %, showing that the activity follows the Phosphorus chemistry.

Equally detailed absorption measurements were not possible in samples S.1 and S.2. The counting rate due to P^{33} in sample S.1 is small. In the case of sample S.2 the P^{33} fraction is significant, but the total counting rate is low.

Analysis of the counting rates at various absorber thicknesses in samples S.1 and S.2 (fig. 2) reveal the same behaviour as that presented in S.3 (fig. 1).

The measured half-lives of the low and high energy groups are summarised in table 2; the half life periods are again in agreement with those of the isotopes P^{33} and P^{32} respectively.

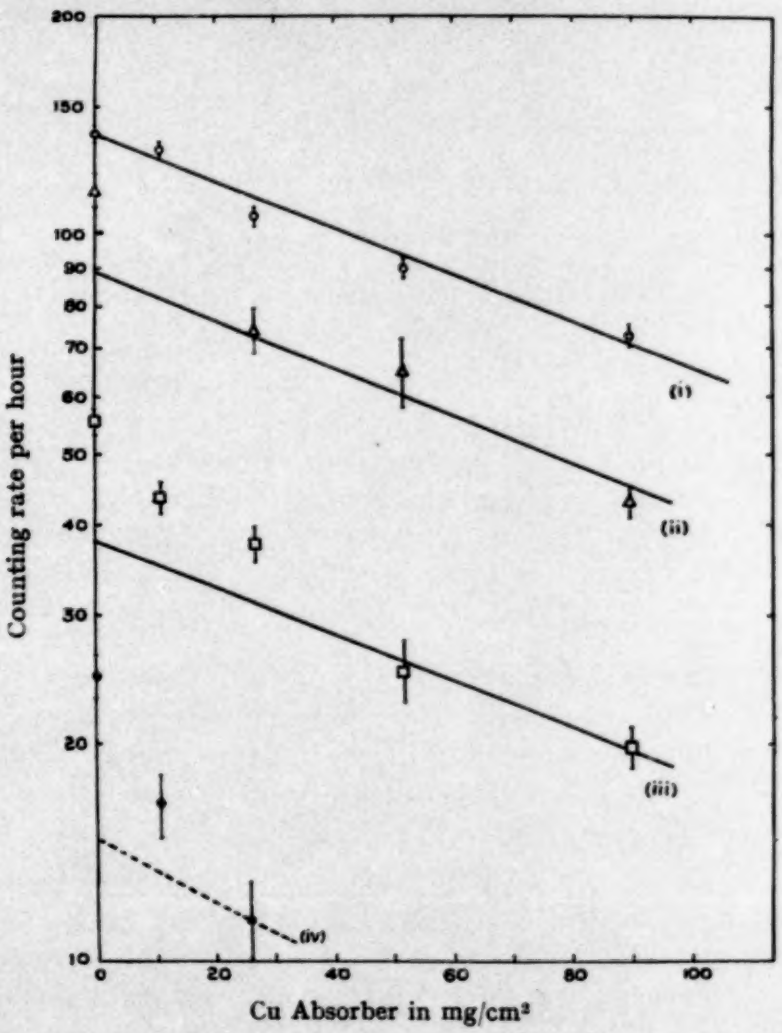

Fig. 2. Counts per hour are plotted against absorber thickness in mg/cm² of copper.
(i): absorption curve for a sample of artificial P^{32}.
(ii) and (iii): absorption curves of rain water sample S.1 taken on two different dates: 16th July, 1956 and 1st August, 1956.
(iv): absorption curve of rain water sample S.2 taken on 3rd August, 1956.

The estimated concentrations of radioactive phosphorus in the three samples (in atoms per cm³ at the time of rainfall) is given in table 2. These estimates are based on a 65 % efficiency for chemical extraction as estimated

from pilot experiments. As mentioned earlier, the chemical efficiencies are not yet well known in our present experiments. Another source of error arises from possible unevenness in the deposition of the material on the sample holder. Additional self absorption due to this cause is negligible for P^{32}, but not for the soft β-radiation from P^{33}.

TABLE 2

Sample No	Low Energy Component (P^{33})		High Energy Component (P^{32})	
	Half life (days)	Concentration (atoms ml^{-1} of rain water)	Half life (days)	Concentration (atoms ml^{-1} of rain water)
S.1	28.6±9.1	50	13.0±0.9	70
S.2	19.5±8.0	30	12.0±3.1	30
S.3	23.4±6.7	15	15.6±1.1	20

We may make a crude estimate of the expected concentrations of phosphorus isotopes by comparing it with the observed concentration of the isotope Be^7 (half-life 53 days) in rain water, whose cosmic ray origin has already been demonstrated [7, 8, 9]).

The probability of finding a Be^7 nucleus in a disintegration of a nitrogen or oxygen nucleus can be roughly estimated from the results of Kellogg [13]) and those of Dickson and Randle [14]), who determined its yield in the bombardment of a carbon target by 90 MeV neutrons and 156 MeV protons respectively. If one extrapolates their results from carbon to nitrogen, one is led to the prediction that about 5 % of all stars produced by cosmic rays in the atmosphere lead to the production of Be^7.

The probability for producing one of the phosphorus isotopes in the disintegration of argon may be estimated from Rudstam's semi-empirical formula [15]) to be of the order of 2 %, although the application of the formula to mass numbers as low as that of A^{40} is somewhat questionable.

The fraction of nuclear disintegrations which occur in atmospheric argon is approximately equal to the ratio of argon to nitrogen and oxygen nuclei in the atmosphere multiplied by the ratio of their geometric cross sections. This fraction is 1 %.

If we assume that both Be^7 and phosphorus isotopes are collected in precipitation with comparable efficiency and that (as is indicated by the presence of the short-lived isotope P^{32}) the time between production and collection is not long compared to the half-lives, we must expect a concentration ratio

$$\frac{P^{32}}{Be^7} \approx \frac{2}{5} \times \frac{1}{100} = 4 \times 10^{-3}.$$

The observed ratio should be smaller; it decreases as the time between production and collection is lengthened since the half life of P^{32} is shorter than that of Be^7.

The average Be^7 concentration [16]) in rainfalls in Bombay during 1956 was $\approx$ 5500 atoms per cm^3 while the phosphorus concentration given in table 2 is about 40 atoms per cm^3. Thus the observed concentration ratio is 7×10^{-3}. Marquez and Costa [1]) observed an average concentration of 6 atoms per cm^3 corresponding to a ratio of 10^{-3}.

In view of the approximations involved in the above estimate, we cannot expect a better agreement and conclude that the observed phosphorus concentration is consistent with its being due to cosmic ray induced disintegrations in the atmospheric argon.

4. Conclusions

Our results show the presence of the two phosphorus isotopes P^{33} and P^{32} in rain water. The conclusion is supported by the following evidence. The phosphorus samples extracted chemically from rain water, show β-activity. Absorption measurements show the presence of two distinct β-groups whose energy and half-lives are the same within statistical errors as those of P^{33} and P^{32} nuclei. A second chemical purification yields a high percentage of recovery.

The concentrations of the two phosphorus isotopes in rain water are not inconsistent with their being produced in the disintegration of atmospheric argon nuclei by cosmic ray particles.

We are grateful to Professor B. Peters for his guidance and keen interest in this work. Our thanks are also due to Mr. R. V. S. Sitarama Rao and Mr. H. L. N. Murty for constructing the counters.

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X-RAY ESCAPE GATING AS APPLIED TO A PROPORTIONAL COUNTER

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Abstract: A method of low-energy γ-spectroscopy is described which employs a xenon-filled proportional counter and a NaI scintillation counter. The xenon counter is placed in the gamma flux while the scintillator counts the xenon fluorescence X rays which follow photoelectric absorption of γ rays in the proportional counter. A pulse from the proportional counter is passed to the kicksorter only when the scintillator receives a xenon K X ray. The recorded spectrum thus contain only K X ray escape peaks and is easy to interpret. The method is described in detail and a practical example is given. A possible extension of the method to conversion-electron spectroscopy is proposed.

1. The Apparatus

One difficulty with gamma spectroscopy by proportional counter is the presence of escape peaks which often make the interpretation of a spectrum ambiguous. Unfortunately elements of high atomic number, which have the desired large photoelectric cross section, tend to de-excite themselves by K X ray emission as well as by ejecting Auger electrons. In fact, with Xe and Kr, the most popular gases, thanks to their high density and gamma stopping power, the Auger effect and fluorescence compete about equally so that the K X ray escape and the full energy peaks are of comparable size. Even with mercury vapor whose fluorescence yield is about 95 % the ambiguity is still a nuisance.

It would be nice to suppress one of the two sets of peaks, and the obvious victim is the full energy set. The method (fig. 1 and 2) is to have a scintillation counter looking at the xenon counter and to send a count from the latter to the kicksorter only when a xenon fluorescence K X ray has been observed to emerge from it. The spectrum of such counts will contain only escape peaks. This method works very well; a variety of factors turn out to work in its favor. For example, if the scintillator were not shielded it would be swamped by the direct gamma flux from the source; however, at proportional-counter energies shielding is easy, so the scintillator will see little but scattered radiation unless the source emits high energy gammas as well. Obviously the same shielding virtually eliminates fluorescence from the sodium iodide crystal. As the irradiated side of the lead shield need not be visible to the xenon counter lead fluorescence is not a problem.

The output of the scintillator is fed to a differential discriminator set to accept pulses of a size appropriate to xenon K X rays. If, however, there is

in the observed spectrum a gamma ray of less than 60 keV, these gammas, Compton-scattered from the counter gas and walls, may open the gate more often than the desired fluorescence X rays. Fortunately most of the Compton processes occur in the walls from which the recoil electrons seldom emerge. The few recoil electrons which are produced in the gas or get into it from the walls will have less than 11 keV, assuming a 60 keV photon. These are unlikely to be confusing. Higher energy photons will produce more energetic electrons but the scattered photons will then be rejected by the differential discriminator, so the counts will not be recorded.

In consequence one gets very thorough suppression of the full energy peaks with little distortion of the remainder of the spectrum. Such distortion as occurs will arise because the scintillator sees the most X rays from events nearest to it. Wall effect may be different in the escape-gated spectrum from that in the total spectrum since the observed events in the two cases will have a different space distribution within the counter. In addition, with very low energy gammas, an appreciable fraction of the beam is absorbed

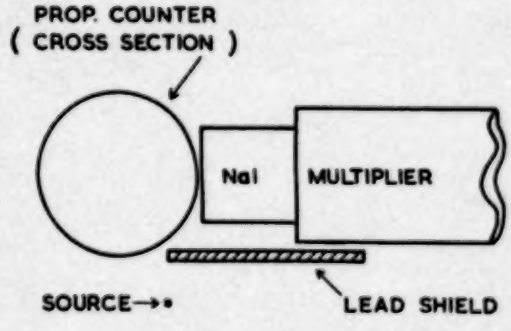

Fig. 1. Physical arrangement.

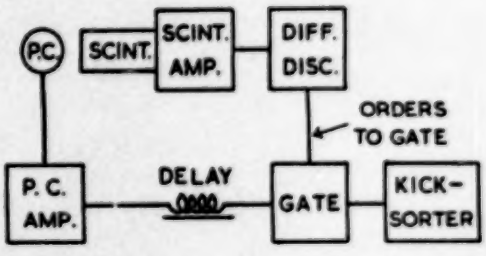

Fig. 2. Electronic logical arrangement.

in the xenon. The space distribution of events will be different for different energies, so the spectrum of gammas causing events close to the scintillator may be different from the average taken over the whole counter.

The counting efficiency can, in principle, be high as each count in an escape peak is accompanied by an X ray. In practice the most serious limitation is the high price of sodium iodide which limits the solid angle of the X ray catcher. With a 1 inch cube of sodium iodide separated from the xenon by 1 mm of Al and 1.5 mm glass (65 % theoretical transmission for normally incident Xenon Kα X rays) but otherwise as close as possible the relative efficiency (the ratio of the gated to the total count rate) is about 0.014.

There is one final snag. Xenon ejects one Kβ X ray for every four or so Kα X rays, so each peak will have a satellite about 4 keV below it. The scintillator resolution is not good enough for clean separation of Kα from

Kβ; however, lowering the differential discriminator window gives some suppression of the satellites at the price of reduced count rate. Two runs with different window bias can thus identify the Kα escape peaks. A critical absorber in front of the scintillator offers little hope of rejecting the Kβ X rays; critical absorbers require good geometry to avoid trouble from fluorescence in the absorber while good coincidence efficiency demands bad geometry. Critical absorption appears in another connection whether we want it or not. Xenon Kβ X rays are above the iodine K edge and penetrate very little into the sodium iodide scintillator crystal. The resulting iodine fluorescence X rays will be likely to escape with the result that the counting efficiency of the crystal is reduced from 100 % to about 75 %. Xenon Kα is below the iodine K edge, so penetration will be deeper and the efficiency should be nearly 100 %. This means that even with the differential discriminator accepting all the Xenon K X rays there will be a 25 % reduction in the ratio of Kβ and Kα escape peaks when escape gating is used.

2. The Spectrum of Am^{241}

By way of conclusion regard what the method does to the spectrum of Am^{241}. This has one principal gamma at 59.7 keV, a weaker gamma at 26.4 keV and some Np L X rays at about 15 keV. Fig. 3 shows the total

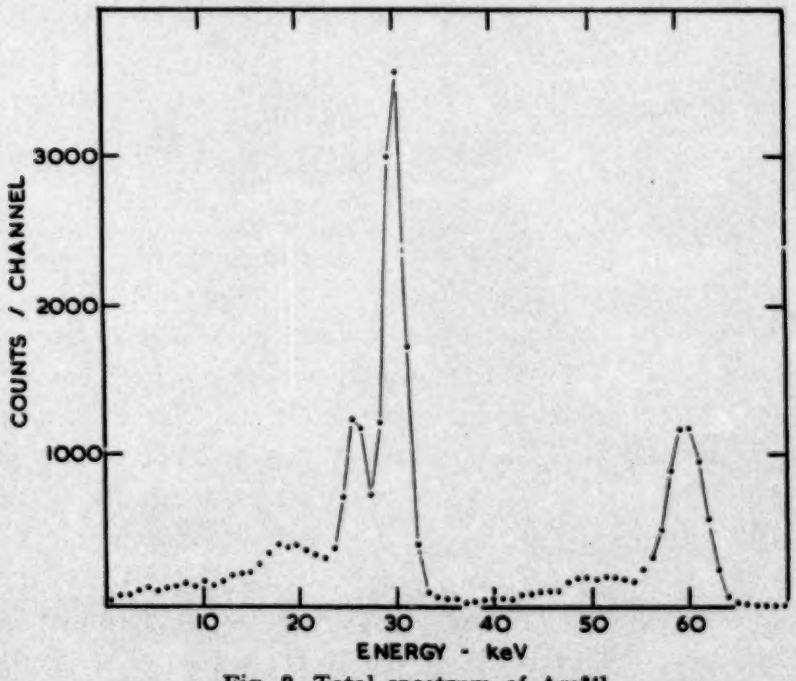

Fig. 3. Total spectrum of Am^{241}.

spectrum from the xenon counter. At 60 keV is the full energy peak while at 30 and 26 keV are the Kα and Kβ escape peaks respectively. The 26 keV peak also contains a contribution from the 26.4 keV gamma ray. The Np L

X rays have been badly attenuated by the material of the source holder and are not visible. This spectrum was recorded in 35 seconds.

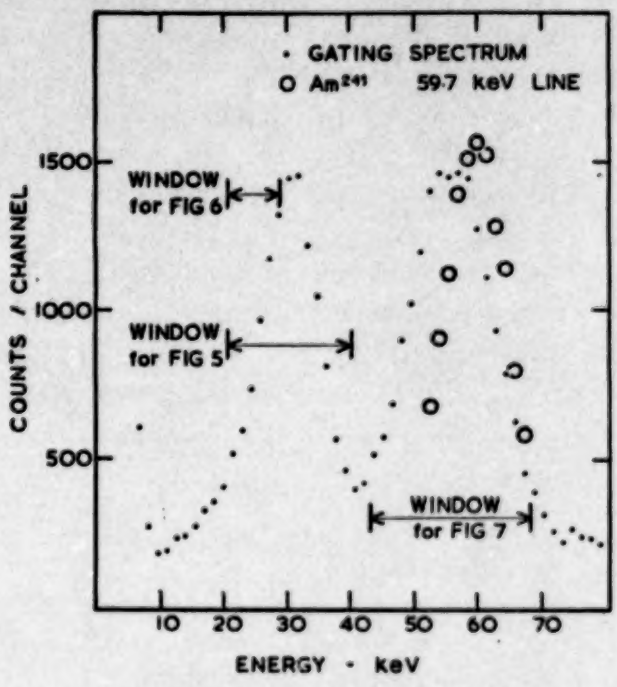

Fig. 4. Gating spectrum from scintillator.

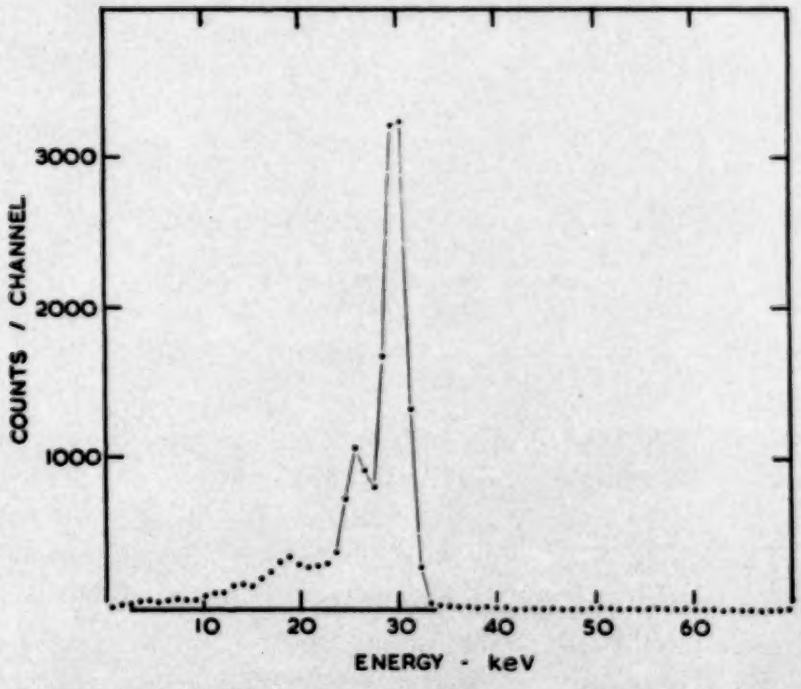

Fig. 5. Escape gated Am^{241} spectrum (see fig. 4 for differential discriminator window).

Fig. 4 shows (in dots) the spectrum of scattered radiation recorded by the scintillator. Xenon K X radiation appears at 30 keV while the Compton

scattered 59.7 keV gammas form the broad upper peak. For comparison, Am^{241} gammas were allowed to fall directly on the crystal (circles). As directly backscattered gammas have an energy of 49 keV the Compton peak appears to be in the right place.

Fig. 5 is the escape-gated spectrum with the differential discriminator accepting the entire X ray peak (see fig. 4). This took 40 minutes to collect, which indicates relative efficiency of 0.014. The unimportance of random coincidences is shown by the nearly perfect suppression of the full energy peak. This is not surprising in view of the fact that each gate interval contained, on the average 0.7 coincidence counts. If more complete suppression of the full energy peak were required the coincidence resolving time could be pared down from the 12 μsec used in these experiments. The

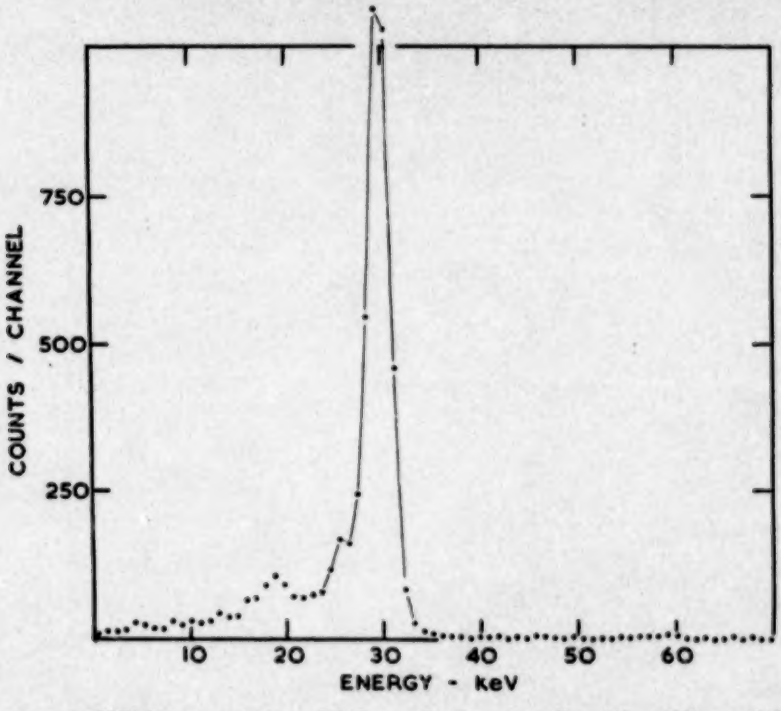

Fig. 6. Escape gated^{241} Am spectrum with low window (see fig. 4 for differential discriminator window).

Kβ escape peak at 26 keV is noticably reduced, partly by the suppression of the 26.4 keV Am^{241} gamma, and partly because as described above, the scintillator discriminates slightly against Kβ.

Fig. 6 shows how a Kβ escape peak may be identified. The differential discriminator was set low (see fig. 4) and the resulting discrimination against Kβ has nearly eliminated the Kβ escape peak. As this run also lasted 40 min the relative efficiency is seen to be down to 0.004.

As it is not always possible to separate Compton-scattered gammas from fluorescence X rays the differential discriminator window was purposely set on the Compton peak to see how many recoil electrons would be counted.

Fig. 7 (another 40 min run) shows that the number is negligible and that they occur at less than 11 keV, as expected. On this run the average number of counts per gate interval was 0.006 which gives a rough idea of the fraction of the Compton events which occur in the counter gas.

Am^{241} was chosen as an example only because the Compton peak and the fluorescence X rays do not overlap in the scintillator spectrum. It does not show the method at its best since all but one of the Am^{241} lines are below the xenon K edge and are suppressed when escape gating is used. The desired spectrum is thus the total minus the escape-gated spectrum. As the relative efficiency is hard to determine accurately and as any such determination may be vitiated by spectral distortion such a subtraction will not mean much. Clearly, this method belongs above the K edge; however, that limitation still leaves it a large range of usefulness.

There is one further point. The X ray escape is prompt and this tells exactly when the photoelectric event occurred in the proportional counter.

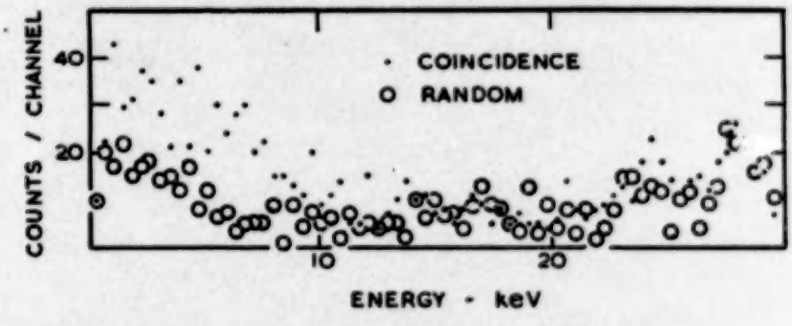

Fig. 7. Compton gated spectrum (see fig. 4 for differential discriminator window).

By using the X ray catcher signal for timing it should be possible to do fast coincidence experiments between the proportional counter and (say) another scintillator. Whether this is as useful as it is amusing must remain to be seen.

3. Possible Application to Conversion Electron Spectroscopy

It is obvious that an analogous method applied to magnetic (or other) spectroscopy of conversion electrons would simplify the identification of K conversion lines. (This may already have been done but I have not heard of it.) When an external radiator is used it should be possible to shield the X ray catching crystal from the direct gamma flux from the source and still to obtain a very high collection efficiency for the fluorescence X rays. Avoiding the direct gamma flux demands that the source and radiator be separated, which means reduced count rate. Reduced light collection will be imposed by the light pipe required to get the photomultiplier out of the magnetic field of the spectrometer, but this will be at least partially com-

pensated by the more energetic X rays resulting from the high Z of the radiator.

When internal conversion is examined the X ray catcher must necessarily be in the direct gamma flux. Fortunately the ratio of conversion electrons to gamma rays is normally higher than with an external radiator. In practice it should be possible to take both K conversion and total spectra at the same time. At the end of one run one should have unambiguous values for the transition energies and the various conversion ratios as well.

A similar scheme has actually been tried for the case of internal conversion, with a large argon-filled proportional counter as the electron spectrometer. A Gd^{153} source, deposited on a thin layer on aluminium, was placed inside the counter next to an aluminium window just outside which was the X ray-detecting scintillator. A spectrum of proportional-counter pulses in coincidence with Eu (the daughter) K X rays showed enhanced K conversion lines. Complete suppression of everything else was not to be expected for as Gd^{153} decays by K capture there will always be Eu K X rays in coincidence with the Eu decay whether or not K conversion occurs. Had a magnetic spectrometer been used instead of the proportional counter the electron counting efficiency would have dropped by a factor of twenty or so. However since the coincidence resolving time could have been reduced by a factor of 100 or more the random coincidence situation appears similar.

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ELECTRIC MONOPOLE TRANSITION RATE IN Zr^{90} †

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Abstract: The mean life of the 1.75 MeV excited state in Zr^{90} with $J = 0$, even parity, has been found to be 8.5 ± 3 mμsec by observing delayed coincidences between beta rays from Y^{90} and annihilation radiation due to internal pair creation. This corresponds to $\varrho^2 = 0.036$, close to the values for the other three known monopole transition probabilities.

1. Introduction

There is currently increased interest in so-called nuclear electric monopole transitions. Such transitions do not give rise to (single) gamma-ray emission but to transfer of the transition energy either to an atomic electron by internal conversion or to internal pair creation. The alternative process of two-photon emission is, generally, negligibly small. The E0 part of the transition field contributes to transitions between states of the same parity and angular momentum. In particular, electromagnetic transitions between states with $J = 0$ and equal parity can proceed only by this process. The interaction takes place entirely inside the nuclear volume, corresponding classically to a radially pulsating spherical charge distribution which does not give rise to a time-varying field outside the charged region. As in the case of internal conversion by higher multipoles, the theoretical transition rate can, in first approximation, be written as a product of terms involving only the nuclear and the electronic wave functions, respectively. Thomas [1]) has given expressions using relativistic electron wave functions which are adequate for all experiments to date. Refinements involving the modification of these wave functions by screening and finite nuclear size [2, 3]), variation of the wave function over the nuclear dimension [2]), electrodynamic corrections [4]) and higher order terms in the matrix element [2]) have been discussed. These may be of interest for transitions of very high or very low energy or in cases where the nuclear matrix element vanishes in lowest order. The precision of available experimental data does not require their inclusion.

The relevant nuclear matrix element is $M = \int \psi_{in}^* \sum_p r_p^2 \psi_{fin}$. The summation extends over all protons in the nucleus. It is convenient [2]) to

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write $M = \varrho R^2$, where R is the nuclear radius; ϱ^2 may be considered a measure of the strength of thet ransition, analogous to the reciprocal of the ft value in beta decay. $(\alpha\varrho)^{-2}$ is analogous to ft. The electron wave functions appearing in Thomas' formulas are tabulated for the calculation of beta spectra. We write the formulas in a form suitable for use of the NBS tabulation [5]). For the rate W_K of K-shell conversion and the rate W_p of pair creation:

$$W_K/\varrho^2 = 7.6\times 10^{14}\,\frac{[S+\sqrt{3}(S+2)]^2}{[\Gamma(3+2S)]^3}(\alpha Z)^{3+2S}(2R)^{4+4S}\,\frac{\varepsilon+1+S}{\eta}\,f_-(\eta, Z)\ \text{sec}^{-1},$$

$$W_p/\varrho^2 = \frac{2.9\times 10^{15}}{\pi[\Gamma(3+2S)]^2}\,(2R)^{4+4S}\int_1^{E-1}\frac{\varepsilon_+\varepsilon_- - (1+S)^2}{\eta_+\eta_-}\,f_+(\eta_+, Z)f_-(\eta_-, Z)\,d\varepsilon_+\ \text{sec}^{-1}; \tag{1}$$

$\varepsilon_+ + \varepsilon_- = E$; $\varepsilon = E+1+S$; $\eta = (\varepsilon^2-1)^{1/2}$; $S = (1-\alpha^2Z^2)^{1/2}-1$; $f_\pm$ is the "Fermi function" as tabulated in ref. [5]); E is the transition energy; relativistic units are used throughout. We use $R = 1.3\times 10^{-13}A^{1/3}$ cm. The decay rate by conversion in all other shells is about fifteen per cent of W_K in most cases [2]). Church and Weneser [2]) have given graphs for W_K for $Z \geqq 40$, using better approximations for the wave functions than are used in eq. (1).

There are four known nuclei with lowest excited states of 0+ character: O^{16} [6]), Ca^{40} [7,8]), Ge^{72} [7]), Zr^{90} [9]). All of them are even; three are "doubly magic", i.e. they contain numbers of both neutrons and protons corresponding to closed shells; the fourth, Ge^{72}, contains forty neutrons corresponding to the filling of the $2p_{1/2}$ shell. The $2p_{1/2}$ shell of *protons* is filled in Zr^{90} and the $1p_{1/2}$ shells for both neutrons and protons in O^{16}. It has been suggested [10]) that the occurrence of a lowest O+ state in doubly magic nuclei may be due to two-particle excitation; for example, excitation of two particles from the $(p_{1/2})^2$ to the $(d_{5/2})^2$ configuration in O^{16} [11]) or to the $(g_{9/2})^2$ configuration in Zr^{90} [12]). A similar explanation has been proposed for the (second) excited 0+ state in C^{12} [11]) and might be thought applicable to Ge^{72}. Since, in an independent particle approximation, electric-monopole (E0) transitions are due to a single-particle operator, a two-particle transition must be due to the perturbation by the interaction between the particles. If the known free two-body interaction is assumed for this perturbation, this model leads [11]) to transition probabilities to the ground states of C^{12} and O^{16} which are 36 times smaller than the observed values. It might just be consistent with the observed life-time in Ge^{72}.

In the light nuclei the theoretical transition probability may be increased by assuming [13, 14]) an appropriate amount of single-particle excitation in the excited configuration. This assumption would not be plausible in the heavier nuclei. Alternatively the transition rate may be increased by increasing the number of particles participating in the nuclear transition, i.e. by as-

suming correlated or collective motion. Reasonable results have been obtained with an alpha-particle model [11, 15]) of O^{16}, but the validity of the assumptions involved is doubtful. Radial oscillations of a compressible nucleus could give rise to E0 transitions, but the nuclear compressibility required to explain both the energy of the 0+ states and transition probabilities is much larger than that deduced from other evidence [11]). Oscillations of an incompressible liquid drop have been considered [2]). Two-phonon excitation may give rise to 0+ excited states with ground-state transition probabilities in the range of interest, but the model never predicts 0+ for the lowest excited state.

The experimental evidence concerning E0 transition probabilities is extremely scanty. Prior to the present investigation only two life-times of $0+ \rightarrow 0+$ transitions had been measured directly, namely those in O^{16} and in Ge^{72}. In addition, the transition probability from the ground state of C^{12} to the 0+ level at 7.68 MeV has been determined by inelastic electron scattering [16]). There exists also an estimate of the ground-state transition rate for the 1.42 MeV 0+ level of Po^{214}, but this is based on a somewhat uncertain estimate of the competing alpha-ray emission. H. Kendall, in this laboratory, has very recently set an upper limit on the decay rate from the 0+ state at 1.2 MeV in Ge^{70} to the ground state. Motz [17]) has found evidence for a 1.3 MeV E0 transition in Cd^{114} formed by neutron

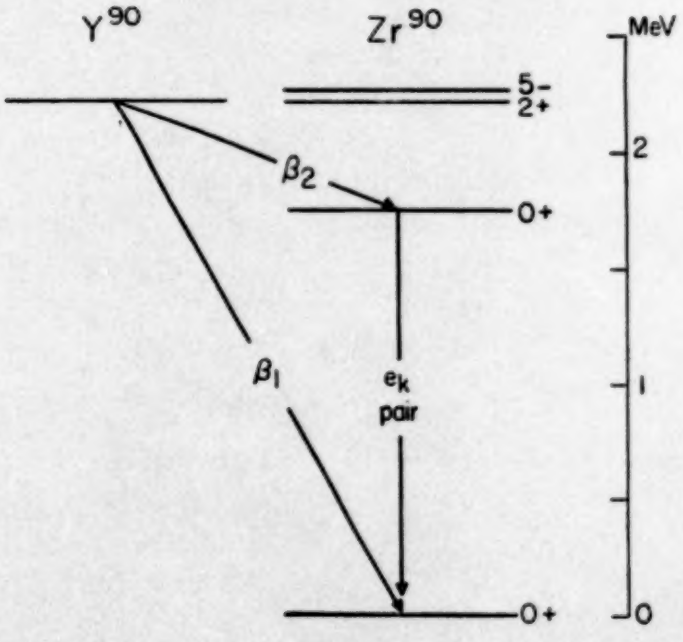

Fig. 1. Energy level scheme for the decay $Y^{90}-Zr^{90}$. The 5− and 2+ states are not reached in this decay.

capture. The rate of this transition can be estimated with certain assumptions concerning the rates of competing gamma-ray transitions to other states.

In a similar fashion, Alburger [18]) has set an upper limit on the transition rate from the 1.25 MeV 0+ state in Pd^{106} to the ground state. Some recent attempts to detect an E0 contribution in the internal conversion of transitions between states of equal but non-zero spin [2]) have yielded a negative result. The experimental and theoretical complications of these cases are, however, considerable.

In view of the paucity of the experimental evidence it seemed worth while to determine the transition probability from the 1.75 MeV 0+ state in Zr^{90}. This transition was discovered by Johnson *et al.* [9]) in the beta decay of Y^{90} (cf. fig. 1). These authors reported 5×10^{-5} internal conversion electrons and $(2\pm1)\times10^{-4}$ positons per Y^{90} decay but less than 5×10^{-6} gamma rays of corresponding energy. Subsequently Greenberg and Deutsch[19] determined the positon intensity as $(3.6\pm0.9)\times10^{-5}$, and Yuasa, Laberrigue and Feuvrais [20]) gave $(1.6\times0.6)\times10^{-4}$ for the number of conversion electrons. The calculation of Thomas yields for the ratio of conversion electrons (from all atomic shells) to the number of pairs: $N_e/N_p = 3$, a value consistent with the experimental results.

Because of the very low abundance of the monopole transition in the decay of Y^{90}, the only convenient mode of excitation, the experiment had to be designed with considerable care. Since the expected life-time was in the millimicrosecond range, the delayed-coincidence method [21]) was indicated. The 0.5 MeV beta-ray spectrum of Y^{90} presents the "preceding" radiation. For the "succeeding" radiation we had the choice between conversion electrons and positons from pair creation. The choice is made with the aim to obtain the highest coincidence rate with a given ratio of true to accidental coincidences. One must consider the abundance of the radiation, the practicable efficiency for its detection, and the ability of the detector to discriminate against other radiations which would contribute accidental or unwanted true coincidences. The balance of these factors indicated that without a superior high-transmission beta-ray spectrometer detection of the positons is the preferable approach.

2. Experimental Method

The experimental arrangement used in our experiments is shown in fig. 2. Sources of Y^{90} were prepared by carrier-free precipitation from Sr^{90} parent, deposited on aluminized "Mylar" foil and covered by another such foil. Calibrating sources of Co^{60}, Na^{22}, and Y^{90}—Na^{22} mixtures were prepared in analogous fashion. The aluminized source "sandwich" was inserted in the 4π detector A consisting of a split plastic (Pilot B) scintillator, shaped to reflect a maximum amount of light to the cathode of the RCA 6810 photomultiplier tube on which it is mounted. This counter served to detect the "preceding" radiation. The dimensions of the plastic are such that all

positons stopped in it and such that all but the most energetic beta rays lost their entire energy in it, permitting pulse height selection for various parts of the beta-ray spectrum.

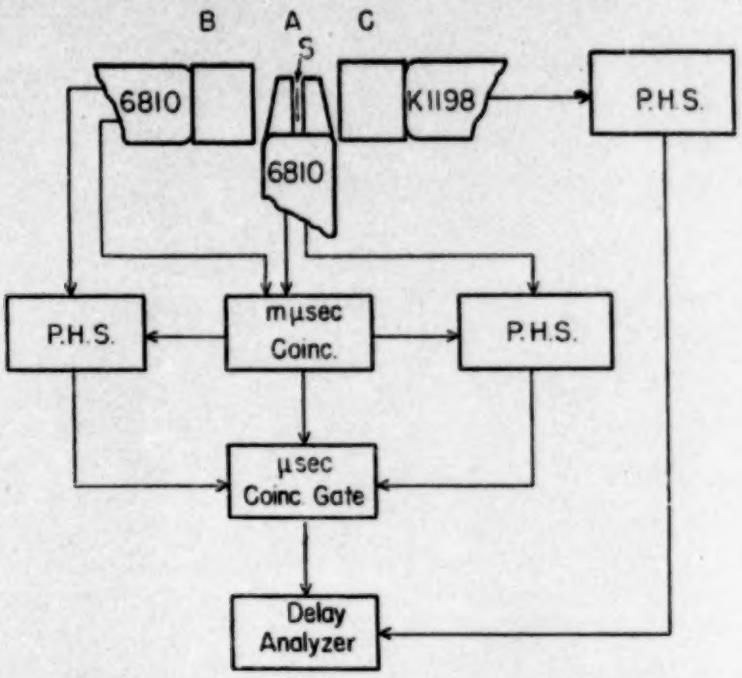

Fig. 2. Experimental arrangement of counters and electromic circuits.

The positons were detected by means of the two annihilation quanta in counters B and C. Counter B is a 2 inch by 2 inch plastic cylinder, mounted on a RCA 6810 tube, counter C a NaI (Tl) crystal 3 in. in diameter, 2 in. thick, mounted on a DuMond K 1198 photomultiplier.

The electronic system is basically of the conventional "fast-slow" type for counters A and B. Counter C, with a single-channel pulse height selector for the annihilation energy, provides an additional "slow" gate for the final output. The fast coincidence circuit is of the Bell-Graham diode-mixer type. The slow coincidence circuit is actuated by the pulses from pulse height selectors in all three channels and opens a linear gate for the pulses from the fast circuit. These gated pulses from the coincidence diode are presented to a five-channel kick-sorter acting as a crude time analyser [22]). In order to reduce the resolving time requirements on the pulse height selectors, linear gates actuated by fast coincidences admit only coincident pulses to the selectors in channels A and B.

The overall efficiency of the system for detecting a positon stopped in counter A is $(5\pm1)\times10^{-3}$, as determined by means of a Na^{22} source. Most of the uncertainty in this number is due to uncertainty concerning the effect of the nuclear gamma-rays emitted in the decay of Na^{22}.

At the beginning of the experiments a search was made for a life-time of some tens of millimicroseconds, expected by analogy with Ge^{72}. The

result was negative and the search was shifted to shorter times. Because of the very low counting rates encountered as well as of certain complications discussed below, no attempt was made to measure the life-time with great

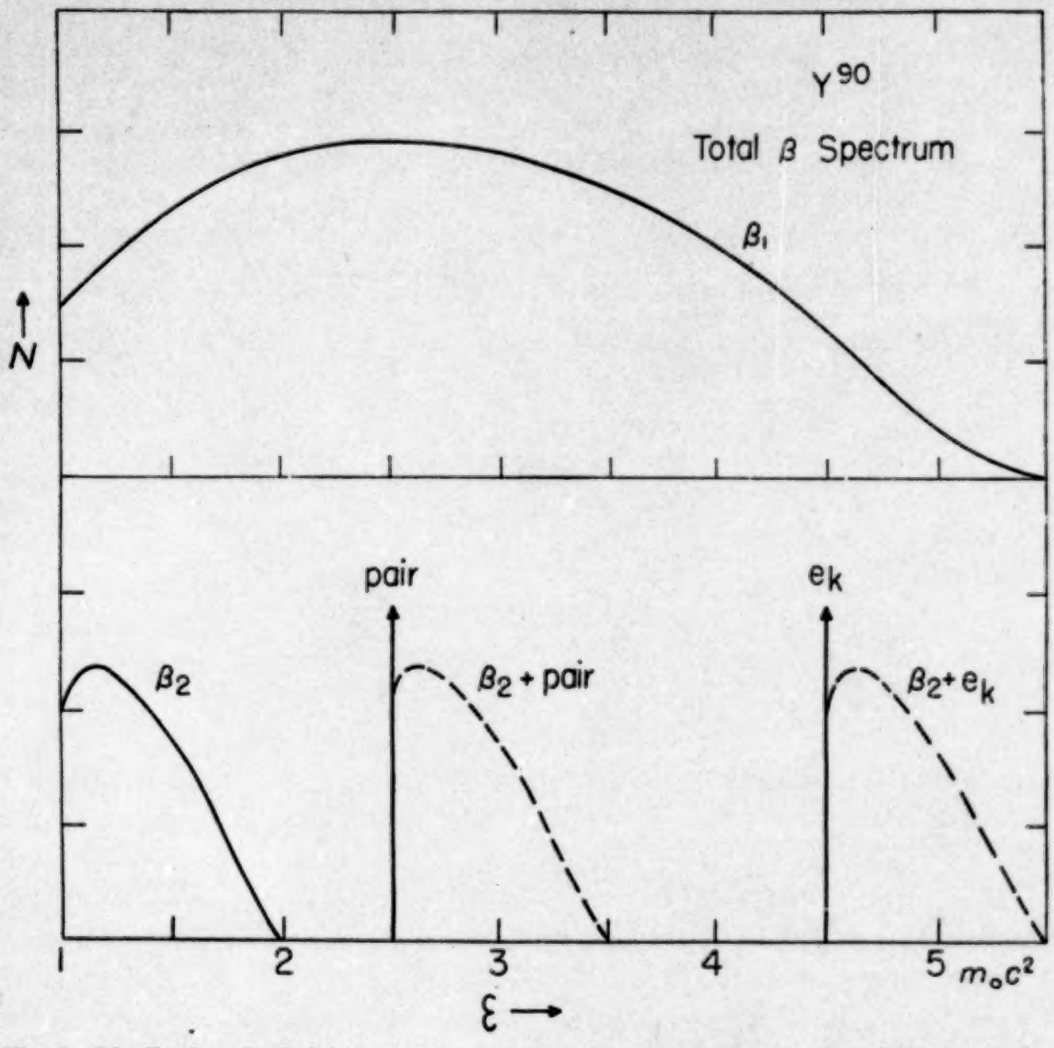

Fig 3. Idealized pulse-height spectra expected from ground-state transitions and from transitions accompanied by internal conversion or pair creation respectively.

accuracy or to obtain data in an easily presentable form. Instead, the effort was concentrated on ascertaining that the observed period in the millimicrosecond range really corresponds to the phenomenon investigated.

The curve of the number of coincidences vs. delay time is not expected to follow a simple exponential decay in its initial part. This is due in part to the nature of the decay process and in part to the operation of the instrument. Consider the energy spectrum of electrons emitted by Y^{90} as illustrated schematically in fig. 3. The main beta-ray spectrum, unaccompanied by any other radiation (except bremsstrahlung) has an end-point at 2.25 MeV (curve marked β_1) and constitutes all but about 2×10^{-4} of the disintegrations. The spectrum leading to the formation of the 1.75

MeV state (curve marked β_2) has a maximum energy of 0.5 MeV. Each of these beta rays is followed within a few millimicroseconds either by a conversion electron of about 1.7 MeV (marked e_K) or by an electron pair with total energy 0.73 MeV. The line marked "pair" is drawn at this energy since this is what will ideally be presented to counter A if both particles enter it. The electronic circuits selecting pulse height in counter A are relatively slow so that the light pulses due to the low energy beta spectrum and to the electrons from the electromagnetic transition will be integrated, giving rise to two displaced spectra (dashed curves in fig. 3). In the case of ideal 4π geometry the dashed curve marked β_2+pair should represent the apparent pulse-height spectrum in counter A corresponding to the preceding radiation. In small solid angle geometry the solid curve would be applicable. In practice the situation is intermediate because of the finite gap between the halves of the scintillator and because of absorption in the source. Nevertheless the dashed curve probably corresponds more nearly to the actual situation.

The time-resolution curve of the coincidence circuit depends somewhat on the pulse height in counter A. The proper pulse height range to be selected in calibrating the circuit is that corresponding to the solid curve marked β_2, even though the range corresponding to the dashed curve is selected in the actual experiment, because the timing is determined by the nuclear beta ray only, regardless of the pulse height due to the electron pair which appears several millimicroseconds later. The appearance of two successive pulses in counter A causes another complication: If the delay between the pulses is longer than the recovery time of the fast pulse shaping circuit, two coincidences will occur; one "prompt" at the time of emission of the positon and one "delayed" corresponding to the time interval between the events. In practice the recovery time is determined by the time required to recharge the grid of the WE404A pulse shaping pentode. Although this depends on the magnitude of the pulse, it is probably always of the order of ten millimicroseconds or longer so that only a relatively small number of "prompt" coincidences will be caused by this effect. Other prompt coincidences will occur due to the imperfection of the 4π geometry, particularly due to absorption of some of the beta rays, and due to occasional events involving bremsstrahlung rather than annihilation radiation.

Establishment of the time-resolution curve is complicated by the finite annihilation life-time of the positons in the plastic A. About two thirds of the positons are annihilated within a few tenths of a millimicrosecond after coming to rest in the plastic, while the remainder, forming positonium, decay with a mean life of about two millimocroseconds [23]). This effect is illustrated in fig. 4. Curve a obtained with a source of Co^{60} represents the resolution curve of one of the time channels; curve b, obtained with a

source of Na^{22}, represents the combined effect of the instrumental time resolution and the decay of the positons. The mean life of the slow component of annihilation in the Pilot B plastic appears to be about 2.5 mμsec.

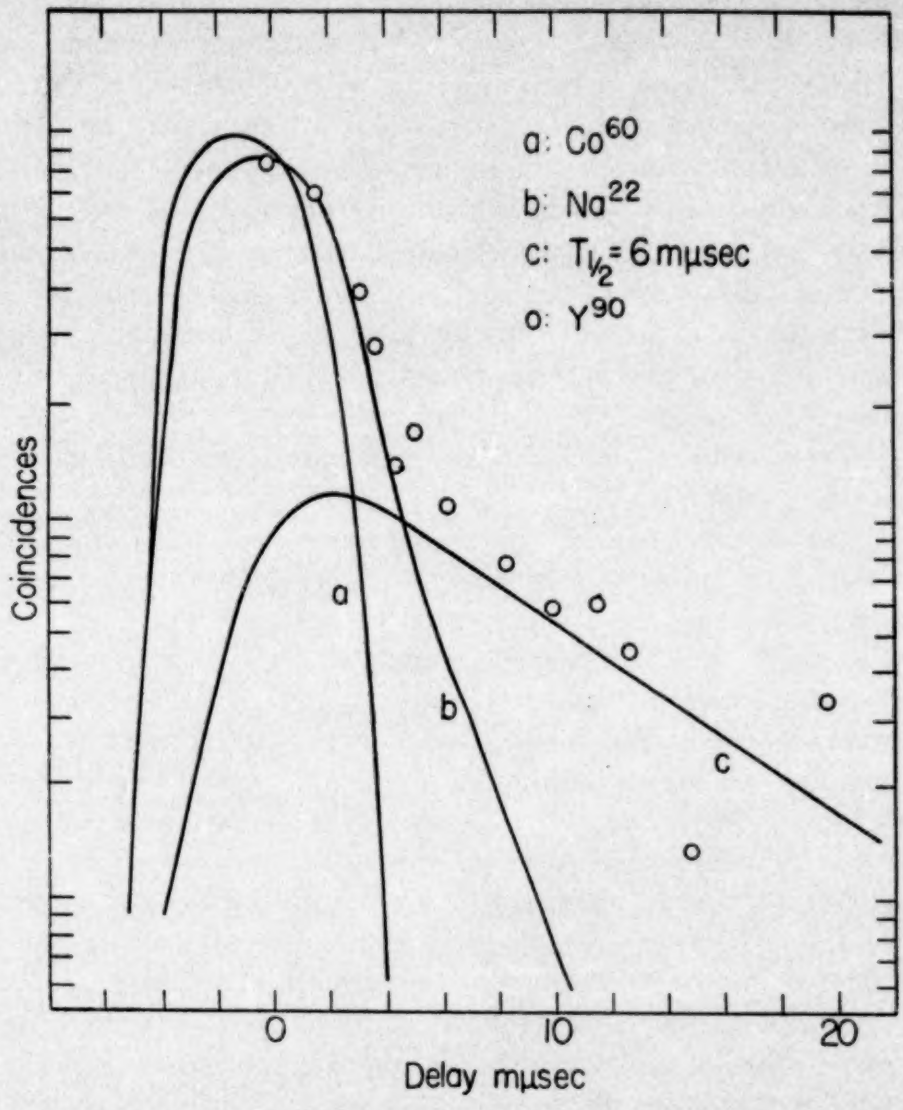

Fig. 4. Delay curves for prompt gamma rays, prompt positons, and postirons from Zr^{90}.

Curve b in fig. 4 may be considered the effective resolution curve for our experiment. In curve c we have folded this resolution curve with an exponential decay of 8.5 mμsec mean life. The results obtained with the Y^{90} source can be fitted with a combination of curves b and c. An additional contribution of prompt coincidences from causes other than positons following curve a cannot be excluded.

3. Results

The coincidence counting rate obtained in this experiment was exceedingly low. In a typical run the several relevant rates were approximately:

A : 5×10^4 sec^{-1}; B : 10 sec^{-1}; C : 15 sec^{-1}; twofold (A—B) fast coincidence rate: 1 sec^{-1}; threefold coincidence rate in each time channel: 2×10^{-3} sec^{-1}. Background and accidental coincidences were completely negligible in all cases for triple coincidences. Because of the very low counting rates involved it was impossible to trace complete decay curves with significant statistical accuracy with a single source. The relative number of prompt and delayed coincidences varied considerably from one source to the other, probably because of varying absorption of the soft beta rays. Since the shapes of the resolution curves of the several time channels differed from each other, it was difficult to normalize points obtained with different channels to the same curve. The mean life of the delayed coincidences was therefore obtained from an average of the slopes of the exponential "tail" measured in the several channels. The points shown in fig. 4 represent an attempt to combine all of the data obtained with one of the sources. The relatively large number of coincidences with delays beyond about 6 $m\mu$sec, compared with curve b, shows clearly the presence of a period in the mμsec range. The average of all our data indicates a mean life $\tau = 8.5\pm3$mμsec. Several measurements were performed to ascertain that the "tail" on the decay curve was not due to malfunctioning of the apparatus under the extreme conditions of the experiment and that the delayed counts were due to positons. The two-fold (A—B) coincidences are primarily due to bremsstrahlung accompanying the main beta-ray spectrum. The delay curve for these coincidences followed closely a, fig. 4. When the pulse-height channel for counter C was varied between 0.3 MeV and 0.7 MeV, the two-fold coincidences showed the expected rapid rise with decreasing gamma-ray energy. The triple coincidences, on the other hand, almost disappeared for energies other than the 0.5 MeV annihilation radiation. When a small amount of Na^{22} was added to a Y^{90} source, the additional triple-coincidence rate followed curve b, fig. 4, indicating proper functioning of the equipment at the counting rates used. From the number of triple coincidences and beta-ray counts from Y^{90}, together with the detection efficiency of 5×10^{-3} mentioned in the preceding section, we deduce the number $(8\pm3)\times10^{-5}$ positons per disintegration. This may be an overestimate if some of the prompt triple coincidences are due to other causes. It is to be compared with the value $(3.6\pm0.9)\times10^{-5}$ found by Greenberg and Deutsch [19]).

4. Conclusion

We can compare the observed decay rate of 0.12×10^9 sec^{-1} with that obtained from eq. (1). Using the value [2]) of 2.2×10^9 sec^{-1} for (W_K/ϱ^2) combined with the value of 9.5 for the ratio of K- to L-conversion and the value 2.6 for the ratio of K-conversion to pair creation calculated from

eq (1), we obtain a total reduced decay rate $(W/\varrho^2) = 3.3\times 10^9$. From this and our experimental result we obtain $\varrho^2 = 0.036\pm 0.014$. This is of the same order of magnitude as the value $\varrho^2 = 0.014$ calculated in the same manner for the 0.69 MeV transition in Ge^{72}, but smaller than the values for the light nuclei (cf. table I). The fact that ϱ^2 is larger for zirconium than for germanium is contrary to the conjecture of Yamaguchi [24]) that the monopole matrix element might be independent of the size of the nucleus.

The relatively close agreement between the reduced transition probabilities in Zr^{90} and Ge^{72} is at least consistent with the suggestion that the 0+ excited state is connected in both cases with the filled shell of 40 particles. The interpretation [12]) of this state as a "pseudo-ground state" of the $(g_{9/2})^2$ configuration gives, however, a vanishing transition probability in first approximation It may be noted, that this discrepancy is also reflected

TABLE I

Transition Probabilities of Zero-Zero Transitions

	Transition Energy MeV	Decay Rate $W(E0)$ sec^{-1}	ϱ^2
C^{12}	7.68	—	0.06
O^{16}	6.06	1.5×10^{10}	0.06
Ge^{70}	1.21	$\lesssim 4\times 10^{6}$	$\lesssim 0.015$
Ge^{72}	0.69	2.3×10^{6}	0.014
Zr^{90}	1.75	1.2×10^{9}	0.036
Pd^{106}	1.14	$\lesssim 10^{8}$?	$\lesssim 0.02$?
Po^{214}	1.41	3×10^{9}?	0.0025?

in the beta decay from Y^{90} to this level which is inhibited only by about a factor of two compared with the ground state transition. Since both the beta decay and the electric monopole transitions involve single-particle operators while the 0+ state can be interpreted on an individual particle model only as a two-particle excitation, one must assume very drastic configuration mixing or an essentially collective nature of the excited state.

It is a pleasure to acknowledge the assistance of E. W. Backofen in the preparation of the sources. Dr. H. W. Kendall has contributed valuable discussions of the problem and suggestions concerning the electronic equipment.

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MANY-BODY FORCES AND THE MANY-BODY PROBLEM

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Abstract: It is a shown that, to a good approximation, many-body forces may be taken into account by the definition of an equivalent two-body force for use in shell model calculations of energy levels.

1. Introduction

Bethe [1]) and Goldstone [2]) have recently given a reformulation of the work of Brueckner and his collaborators [3, 4]) on the quantal many-body problem and its relation to the shell-model of nuclear structure. They consider only two-body forces between nucleons and it is usually stated that this whole approach depends on three-body forces being negligible. The purpose of this note is to point out that quantities such as the energy, which we may expect to be well predicted by a shell model, may still be calculated using this approach even if many-body forces are present. This is achieved by incorporating the principal effect of these forces in an effective two-body force.

2. The Effective Two-Body Force

For simplicity we shall consider the case in which two-body and three-body forces only are present. In the spirit of the shell model a single-body potential V is introduced. V will be determined by a self-consistency condition given below. The Hamiltonian for A nucleons is given by

$$H = H_0 + H_I \tag{1}$$

where

$$H_0 = \sum_{i=1}^{A} (T_i + V_i), \tag{2}$$

$$H_I = -\sum_{i=1}^{A} V_i + \sum_{i<j}^{A} u_{ij} + \sum_{i<j<k}^{A} v_{ijk}. \tag{3}$$

T_i is the kinetic energy operator for the ith nucleon; u_{ij} is the two-body force between the ith and jth nucleons; v_{ijk} is the three-body force between the ith, jth and kth nucleons. The eigenfunctions and eigenvalues of the single nucleon equation

$$(T + V - E_n)\phi_n = 0 \tag{4}$$

are assumed known. Shell model wave functions are constructed from these by taking Slater determinants. We shall assume the lowest energy state obtained in this way to be non-degenerate (closed shell case). The wave functions ϕ_n appearing in it correspond to single nucleon states that we shall call passive and dashed suffices will be reserved to apply to these states. The remaining states will be termed excited. In the expansion of the true ground state wave function in terms of these Slater determinants the coefficient of the state in which the passive states $\phi_{n_1'}, \phi_{n_2'}, \ldots$ are replaced by the excited states $\phi_{n_1}, \phi_{n_2}, \ldots$, respectively, will be denoted by $\lambda(n_1, n_2, \ldots; n_1', n_2', \ldots)$. Care must be taken in summing over these suffices to ensure that each distinct state is counted only once.

The energy difference ΔE between the true ground state energy and the energy of the lowest Slater determinant is given by

$$\Delta E = \sum_{n'} \langle n'|W|n'\rangle + \sum_{\substack{n_1 \\ n_1'}} \lambda(n_1;\, n_1')\langle n_1'|W|n_1\rangle$$

$$+ \sum_{\substack{n_1 n_2 \\ n_1' n_2'}} \lambda(n_1, n_2; n_1', n_2')\{\langle n_1', n_2'|u'|n_1, n_2\rangle + \sum_{n'} \langle n_1', n_2', n'|v'|n_1, n_2, n'\rangle\} \quad (5)$$

$$+ \sum_{\substack{n_1 n_2 n_3 \\ n_1' n_2' n_3'}} \lambda(n_1, n_2, n_3;\, n_1', n_2', n_3')\langle n_1', n_2', n_3'|v'|n_1, n_2, n_3\rangle;$$

$$\langle p|W|q\rangle = -\langle p|V|q\rangle + \sum_{n'} \langle p, n'|u'|q, n'\rangle + \sum_{n'n''} \langle p, n', n''|v'|q, n', n''\rangle. \quad (6)$$

In eq. (5) the sums over undashed suffices run over excited states, and the sums over dashed suffices run over passive states, in such a way that distinct states are counted only once. For example $\lambda(n_1, n_2;\, n_1', n_2')$ and $\lambda(n_2, n_1; n_1', n_2')$ refer to the same state so that both must not appear. The matrix elements that appear in (5) are defined by

$$\langle n_1, n_2|u'|n_1', n_2'\rangle = \langle n_1, n_2|u|n_1', n_2'\rangle - \langle n_2, n_1|u|n_1', n_2'\rangle, \quad (7)$$

$$\langle n_1, n_2, n_3|v'|n_1', n_2', n_3\rangle' = \sum_{\mathrm{P}} \varepsilon(\mathrm{P})\, \langle n_1, n_2, n_3|v|n_1', n_2'\, n_3'\rangle. \quad (8)$$

The sum in (7) is taken over all permutations P of $n_1 n_2 n_3$ and $\varepsilon(\mathrm{P})$ is the signature of P. The matrix elements of V, u and v are the matrix elements for scattering from and into the indicated states. Eq. (5) may be proved by adapting Goldstone's treatment [2]) to this case or by simply using a Tamm-Dancoff solution of the equations. †

Although individual $\lambda(n_1, n_2;\, n_1', n_2')$ are certainly not small, Bethe [1]) has given arguments for showing that the third sum in eq. (5) is small for the closed shell case we are considering. *A fortiori* these arguments imply

† All of Goldstone's results may be rederived by this latter method but it is excessively cumbersome.

that the fourth sum is small and we shall neglect it. Then let us define an effective two-body force U by

$$\langle n_1, n_2|U|n_1', n_2'\rangle = \langle n_1, n_2|u'|n_1', n_2'\rangle + \sum_{n'} \langle n_1, n_2, n'|V'|n_1', n_2', n'\rangle. \quad (9)$$

The self-consistency condition is

$$\langle p|V|q\rangle = \sum_{n'} \langle p, n'|U|q, n'\rangle. \quad (10)$$

for all states (excited or passive) ϕ_p, ϕ_q. Then equation (5) reduces to

$$\Delta E = \sum_{\substack{n_1 n_2 \\ n_1' n_2'}} \lambda(n_1, n_2; n_1', n_2')\langle n_1', n_2'|U|n_1, n_2\rangle, \quad (11)$$

which is the same form as the equation for a two-body force U. Of course the $\lambda(n_1, n_2; n_1', n_2')$ are not quite the same as those given by U but we can easily see that they are the same to the approximation used in neglecting the fourth sum.

3. Conclusion

In the approximation obtained by neglecting the fourth sum in eq. (5) all effects of three-body forces may be taken into account by performing a two-body force calculation with the effective force given by eq. (9). The price of this is the increased complexity of the self-consistency condition expressed by eq. (10).

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ON THE UNIVERSAL INTERACTIONS II
K-meson decays into pions

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Abstract: The idea of the previous work which aims at explaining the complicated interactions between elementary particles in terms of universal interactions is further explored, concentrating on the weak interactions between baryon-meson systems. Present experimental situations are analysed qualitatively by the simple phase-space arguments which suggest that it is more sensible to compare the life-times of Λ^0, θ^0 and τ^+-decays than to compare those of θ^0 and θ^+-decay (§§ 2 and 3). Taking $\Lambda^0 \to \mathcal{N}+\pi$ interaction as the main source of the weak baryon-meson interactions, the required property of this interaction in isobaric spin space is investigated in § 4. As an orientation regarding the order of magnitude, the evaluations of the transition probabilities of θ-decays and τ-decays in our model are made in § 5. The interactions which are responsible for the occurrence of the $\theta^+ \to \pi^+ + \pi^0$ are investigated in § 6. The connections with the compound particle models are briefly discussed.

1. Introduction

The previous work [1]) †† has been devoted to understanding complicated reactions between elementary particles, in terms of certain kinds of universal interactions which act between baryon-meson-lepton-photon systems. They are:

(i) *e*-int., electromagnetic interaction;
(ii) *G*-int., strong baryon-meson interaction;
(iii) *f*-int., weak Fermi interaction;
(iv) *g*-int., weak boson-fermion interaction.

It is also pointed out that there seems to remain still the possibility that the *g*-int. can be eliminated by the intermediary of the other three interactions (*G*, *f* and *e*-int.), provided that *f*-int. is allowed to have certain more complicated properties. In other words, *f*-int. might be the source of all weak interactions. In I we did not touch, in detail, the problem of the elementarity of the weak boson-boson interactions such as $\theta^0 \to \pi^+ + \pi^-$, $\theta^+ \to \pi^+ + \pi^0$, $\tau \to \pi + \pi + \pi$ etc. Of course, there is no definite *a priori* reason to deny the existence of the universal boson-boson interaction which may comprise the above reactions. This standpoint—elementary K-meson decay interactions into pions—will be natural, especially in the Goldhaber [2]) model in which only the nucleon, the π-meson and the K-meson are taken as the elementary particles. Here, however, with the object of decreasing

† On leave of absence from Institute of Theoretical Physics, Kanazawa University.
†† Ref. [1]) is cited as I in this paper. Earlier related papers are also cited there.

the number of the universal interactions, we investigate in detail the standpoint taken in I that these reactions are not truly elementary and are derived from the processes involving an elementary transition due to the above mentioned universal interactions as a virtual intermediate step. It will be instructive to note here that $\pi^0 \to 2\gamma$ may be interpreted by the virtual processes due to the baryon-meson interaction and the electromagnetic interaction without introducing new direct interactions between the π^0 and the photon. Our following arguments are based on the assumption that the K-meson decays into pions are not truly elementary processes, whereas Λ^0-decays have more primary properties. This standpoint may find more or reasonable basis in a model proposed by Sakata and by Markov [3]). There, contrary to Goldhaber's model, nucleons and Λ^0's (which are the source of the strangeness quantum number) are treated as the fundamental particles, and other baryons and bosons are regarded as the composite states of them.

2. Experimental Features of some K-meson Phenomena

Here we briefly summarise the important features of the K-meson phenomena † and their theoretical aspects which are closely connected with our following arguments. Many K-mesons ($K_{\pi 2}$, $K_{\mu 2}$, $K_{\pi 3}$, K_{e3}) have been found. The most striking results are that they have very approximately the same masses and the same life-times. Most probably, some of them are different decay modes of the same particle. It has been pointed out in I that this is possible on the assumption of the universal interactions. Dalitz' argument [4]) seems to suggest that $K_{\pi 2}(\theta)$ and $K_{\pi 3}(\tau)$ are not the same particle. There is no strong evidence contrary to the simplest spin and parity assignments; $\theta(0^+)$ and $\tau(0^-)$.

More specifically:

(1) The life-time of $\theta^0 \to \pi^+ + \pi^-$ is

$$\tau(\theta^0 \to \pi^+ + \pi^-) = (1.7\ {}^{+0.6}_{-0.4}) \times 10^{-10} \text{ sec.}$$

Evidence which seems to favour the mode $\theta^0 \to \pi^0 + \pi^0$ with nearly the same life-time as $\theta^0 \to \pi^+ + \pi^-$ has been reported by Osher and Moyer [5]), which implies that θ^0 has even space parity. Also preliminary experiments have given the branching ratio of the above two decay modes [6]). Because of the existence of two θ^0-mesons θ_1^0, θ_2^0 (θ_2^0 must have different decay modes from θ_1^0, which has $\theta_1^0 \to \pi^+ + \pi^-$ and possibly $\theta_1^0 \to \pi^0 + \pi^0$) and possibly of the τ^0 which is the neutral counterpart of the τ^+, definite conclusions cannot be arrived at as yet.

(2) The $\theta^+ \to \pi^+ + \pi^0$ has a life-time of the order $\approx 10^{-8}$ sec which is approximately 100 times longer than that of θ^0.

† Most of the experimental results used in this paper are the latest reported at the Sixth Rochester Conference (1956).

(3) $\tau^+ \to \pi^+ + \pi^+ + \pi^-$ seems to have nearly the same life-time as $\theta^+ \to \pi^+ + \pi^0$.

(4) The frequency ratio of the alternative decay mode of τ^+, $\tau^+ \to \pi^+ + \pi^0 + \pi^0$ is reported to be

$$R(\tau^+) = W(\tau^+ \to \pi^+ + \pi^0 + \pi^0)/W(\tau^+ \to \pi^+ + \pi^+ + \pi^-) \approx 1/4.$$

(5) There is some evidence that the mode $\tau^0 \to \pi^+ + \pi^- + \pi^0$ exists.

The difference between the life-times of θ^0 and θ^+ is striking. Taking θ-decay interaction as an elementary interaction, Wentzel [7]) and Takeda[8]) have shown that it will lead to interesting branching ratios of the various modes of hyperon decays, which are supposed to occur by the intermediation of G-int. and the postulated elementary θ-decay interaction. One remarkable consequence [7]) is that the life-time of Σ^- is expected to be considerably longer than that of Σ^+ because the difference between the life-times of θ^0 and θ^+ will be reflected in the Σ^+ and Σ^--decays. At present, though the Σ^- seems to have longer life-time than Σ^+, the discrepancy does not seem to be so remarkable as in the θ-decay cases. (Steinberger's results [6]) are $\tau(\Sigma^-) = (1.4\ ^{+1.6}_{-0.5}) \times 10^{-10}$ sec which seems to be longer than that of Σ^+ by a factor 5 or 10.) Two attempts have been made, one by Lee and Orear [9]) and the other by Weinstein [10]), to explain the apparent equality of the mean lives of the θ^+ and τ^+-decay by assuming that the θ-meson and τ-meson are two different particles. However, if θ and τ are 0^+ and 0^- respectively, neither of these theories seems to be successful. The Lee-Orear theory, which assumes a γ-transition between θ and τ, requires a larger mass difference $\approx$ 10 MeV between them. Also the direct experiments to find the γ-rays seem to show a negative result [11]). Weinstein, on the other hand, suggests a transition between θ and τ-states in the presence of atomic electric and magnetic fields. According to Salam †, the transition rate is too small for the spin zero case. The purpose of this paper is to explain the remarkable facts (1) to (5) by the universal interactions postulated in § 1.

3. Qualitative Features of the K-meson Decays into Pions

3.1. PHASE SPACE ARGUMENTS

Before entering into the detailed discussion it will be instructive to investigate the qualitative feature of the K-meson decays. In I, it is inferred that the weakness of weak interactions may originate from the same source, say, from f-int. There the comparison of the transition rates of certain kinds of processes was made. By this procedure, we were able to get some conclusions which are free from the complicated structures of the inter-

† A. Salam (to be published). He has also developed brief arguments which are related to our discussion in (3.1). I am indebted to Professor L. Rosenfeld for showing me a preprint.

vening interactions. The following arguments based on the simple phase-space volume analysis will also clarify the situation. The transition probability per second of some process, W, can be written in the form ($\hbar=c=1$)

$$W = \iint (2\pi/\Pi_i Q_i)|M|^2 \varrho_F, \tag{1}$$

where

M is the matrix-element of the reaction;

$Q_i = 2E_i$ for boson and $= 2E_i/m_i$ for fermion, where E_i and m_i are the energy and mass of the i-th particle in and out, respectively †;

ϱ_F is the density of final states per unit energy;

$\iint$ denotes integration over final states.

Let us rewrite (1) in the form

$$\begin{aligned} W &= \iint (c/r_0)|M|^2 \varrho \\ \varrho &= (2\pi/\Pi_i Q_i)(r_0/c)\varrho_F \end{aligned} \tag{2}$$

where r_0 is a quantity with the dimension of a length so as to make $|M|^2\varrho$ dimensionless. Assuming that all Dirac matrices appearing in M are 1 and that M is constant in the integration over the final density, we replace $|M|^2$ by the average value $\langle\overline{M}^2\rangle$. Then, because we extract the factor $\Pi_i Q_i$, $\langle\overline{M}^2\rangle$ can be regarded as the square of the effective coupling constant of the process under consideration. $\langle\overline{M}^2\rangle$ can be evaluated from the observable value W and the easily calculated value of ϱ. The dimensions of ϱ and M depend on the number and the nature of the particles in and out.

TABLE I

Decay modes	Experimental life-times (sec)	$\langle\overline{M}^2\rangle \times 10^{-14}$
$\Lambda^0 \to \mathrm{P}+\pi^-$	3×10^{-10}	3.7
$\pi \to \mu+\nu$	2.5×10^{-8}	0.71
$\theta^0 \to \pi^+ + \pi^-$	1.5×10^{-10}	4.8
$\theta^+ \to \pi^+ + \pi^0$	1.3×10^{-8}	0.055
$\tau^+ \to \pi^+ + \pi^+ + \pi^-$	1.3×10^{-8}	1.5
$\mathrm{N} \to \mathrm{P}+\mathrm{e}+\nu$	7.2×10^{2}	2.7
$\mu \to \mathrm{e}+\nu+\nu$	2.2×10^{-6}	4.2

If we follow the procedure taken by Ogawa [12]) which, in addition to $\hbar = c = 1$, chooses the unit of length as $r_0 = 3\times 10^{-13}$ cm (which means that $c/r_0 = 10^{23}$ sec^{-1}), the evaluated values of the $\langle\overline{M}^2\rangle$ for the various reactions all fall into the same range, within a factor 10, except for the $\theta \to \pi^+ + \pi^0$. Typical examples are listed in table I. It will not be far from the truth to take these facts as an indication that weak reactions known

† The factor $\Pi_i Q_i$ appears because of the normalisation of the wave function so that the matrix-element M is a relativistic invariant.

so far have the same origin. For example, the value of the universal Fermi coupling constant $f(\approx 10^{-49}$ erg cm$^3)$ turns out, in the above units ($\hbar = c = r_0 = 1$) to be

$$f^2 \approx 10^{-14}. \tag{3}$$

Thus from the values of $\langle\overline{M^2}\rangle$ of table I, if in $\langle M^2\rangle$ the contributions from the strong baryon-meson interactions are of the order 1 in our units ($\hbar = c = r_0 = 1$), it is possible to infer that the universal Fermi coupling constant f is able to be the source of the weak interactions.

Thus it is seen that the phase-space arguments are consistent with the interpretation that the reactions $\Lambda^0 \to \mathcal{N} + \pi$, $\theta^0 \to \pi^+ + \pi^-$ and $\tau^+ \to \pi^+ + \pi^+ + \pi^-$ may be explained from the same source of weak interaction, provided the contributions of the strong baryon-meson interactions do not change drastically from one reaction to another in the units $\hbar = c = r_0 = 1$. So the following standpoints can be taken:

(a) The $\Lambda^0 \to \mathcal{N} + \pi$ interaction has a claim to be the main weak link for the θ^0-decay and the τ^+-decay †.

(b) It will perhaps be more realistic to assume that the mechanism responsible for the small experimental value of the life-time of $\theta^0 \to \pi^+ + \pi^-$ is also valid for $\tau^+ \to \pi^+ + \pi^+ + \pi^-$ but fails for the $\theta^+ \to \pi^+ + \pi^0$.

3.2. TRANSFORMATION PROPERTIES OF THE $K_{\pi 2}$-DECAY IN ISOBARIC SPIN SPACE ††

In Pais, Gell-Mann and Nishijima's theory [13]), the θ^+ and θ^0 are regarded as being the different states ($I_3 = \frac{1}{2}$ and $I_3 = -\frac{1}{2}$ respectively) of the same particle of isobaric spin $I = \frac{1}{2}$. The τ^+ and τ^0 are also assumed to form an isobaric spin doublet ($\tau^+ : I_3 = \frac{1}{2}$, and $\tau^0 : I_3 = -\frac{1}{2}$). From this standpoint we cannot expect any difference between the θ^+ and the θ^0-decay from the covariance properties (in ordinary space) of the intervening interactions. The only freedom left may be the transformation property in isobaric spin space of these weak interactions which are supposed to be charge-dependent. It has been pointed out that the difference of the third component of the isobaric spin ΔI_3 takes only the value $\pm\frac{1}{2}$ for all decays so far observed among baryon-meson systems. The results of the more specific and stringent hypothesis, $\Delta I = \pm\frac{1}{2}$, or the requirement that the weak baryon-meson interactions must transform in charge space as the component of a spherical tensor of rank $\frac{1}{2}$ which satisfies charge conservation, have been discussed by many authors [14]). Here we work under weaker restrictions $\Delta I_3 = \pm\frac{1}{2}$. The $\theta \to \pi + \pi$ is consistent with $\Delta I = \pm\frac{1}{2}, \pm\frac{3}{2}$ and $\Delta I = \pm\frac{5}{2}$ according

† Note that for the $\Lambda^0 \to \mathcal{N} + \pi$, $\pi \to \mu + \nu$ and $\tau^+ \to \pi^+ + \pi^+ + \pi^-$ the results listed in table I are free from the choice of the value or r_0 because M has the same dimension for these processes.

†† *Note added in proof:* Recently Dalitz [39]) has also discussed these properties in detail.

as the total isobaric spin of final states is $I = (0, 1)$, $(1, 2)$ or $I = 2$. That is, the S-matrix of this decay in the isobaric spin space has the following form:

for $\Delta I = \pm\frac{1}{2}$:

$$S = I_f(0)\,\theta^{\dot{0}}(\pi^-\pi^+ + \pi^0\pi^0 + \pi^+\pi^-) + I_f(1)\{\sqrt{2}\,\theta^{\dot{+}}(\pi^0\pi^+ - \pi^+\pi^0) + \theta^{\dot{0}}(\pi^-\pi^+ - \pi^+\pi^-)\}; \tag{4}$$

for $\Delta I = \pm\frac{3}{2}$:

$$S' = I_f'(1)\{\theta^{\dot{+}}(\pi^0\pi^+ - \pi^+\pi^0) - \sqrt{2}\,\theta^{\dot{0}}(\pi^-\pi^+ - \pi^+\pi^-)\} + I_f'(2)\{-\sqrt{\tfrac{3}{2}}\,\theta^{\dot{+}}(\pi^0\pi^+ + \pi^+\pi^0) + \sqrt{\tfrac{1}{3}}\,\theta^{\dot{0}}(-\pi^-\pi^+ + 2\pi^0\pi^0 - \pi^+\pi^-)\}; \tag{5}$$

for $\Delta I = \pm\frac{5}{2}$:

$$S = I_f''(2)\{\theta^{\dot{+}}(\pi^0\pi^+ + \pi^+\pi^0) + \sqrt{\tfrac{1}{2}}\,\theta^{\dot{0}}(-\pi^-\pi^+ + 2\pi^0\pi^0 - \pi^+\pi^-)\}; \tag{5'}$$

here $I_f(1)$, for example, denotes the amplitude of the final state with total isobaric spin 1. According to the even or odd space parity of θ the final state of two pions is required to be symmetric or anti-symmetric in isobaric spin space. Thus from (4), (5) and (5'), in order to realise our programme (3.1(b)), it is easily seen that for the θ of even space parity the only way † to make the $\theta^+ \to \pi^+ + \pi^0$ much less probable than $\theta^0 \to \pi^+ + \pi^-$ is to adopt $\Delta I = \pm\frac{1}{2}$ (where $\theta^+ \to \pi^+ + \pi^0$ is completely forbidden) and add smaller components of $\Delta I = \pm\frac{3}{2}$ or $\pm\frac{5}{2}$ ($\theta^+ \to \pi^+ + \pi^0$ can occur in the state $I_f(2)$). For the θ of odd parity, θ^0- decay and θ^+-decay have comparable frequency unless we adopt more complicated structures.

4. Properties of the Basic Weak Interactions $\Lambda \to N + \pi$

As regards the transformation properties of the $\Lambda \to N + \pi$ interaction in isobaric spin space, there are two independent types which lead to the final states of total isobaric spin $I = \frac{1}{2}(\Delta I = \pm\frac{1}{2})$ and $I = \frac{3}{2}(\Delta I = \pm\frac{3}{2})$. The interaction Hamiltonian density is written for each case as follows ††:

for $\Delta I = \pm\frac{1}{2}$:

$$\alpha(\sqrt{2}\,\bar{\psi}_P O_j \psi_{\Lambda^0} K_j \varphi_{\pi^-}^* - \bar{\psi}_N O_j \psi_{\Lambda^0} K_j \varphi_{\pi^0}^*) + \text{c.c.}; \tag{6}$$

for $\Delta I = \pm\frac{3}{2}$:

$$\beta(\bar{\psi}_P O_j \psi_{\Lambda^0} K_j \varphi_{\pi^-}^* + \sqrt{2}\,\bar{\psi}_N O_j \psi_{\Lambda^0} K_j \varphi_{\pi^0}^*) + \text{c.c.}. \tag{7}$$

The mixing ratio $Z = \beta/\alpha$ of the two final states, assuming the same covariance properties in ordinary space for (6) and (7), is determined by the branching ratio of the two alternative decay modes

† See, however, also § 5.3.

†† For instance, ψ_Λ and $\bar{\psi}_\Lambda$ $(= \psi_\Lambda^* \gamma_4)$ denote the destruction and creation operator of Λ^0 respectively. O_j are the Dirac matrices which contract with the operator K_j.

$$R(\Lambda^0) \equiv W(\Lambda^0 \to \mathrm{N}+\pi^0)/W(\Lambda^0 \to \mathrm{P}+\pi^-) = \frac{2Z^2-2\sqrt{2}\,Z+1}{Z^2+2\sqrt{2}\,Z+2}. \tag{8}$$

For the simplest pure cases, we get

$$\text{for } \Delta I = \pm\tfrac{1}{2}\ (\beta = 0), \qquad R(\Lambda^0) = \tfrac{1}{2};$$
$$\text{for } \Delta I = \pm\tfrac{3}{2}\ (\alpha = 0), \qquad R(\Lambda^0) = 2.$$

As regards to the space parity of Λ^0, relative to the nucleons, there is so far no direct determination. From the analysis of the Λ^0–$\mathcal{N}$ force which allows the Λ^0 to be bound in hyperfragments, Wentzel [15]) has concluded that certain assignments of spin and parity of the Λ^0 and θ†,

$$(S_\Lambda,\ S_\theta) = (\tfrac{1}{2}^+,\ 0^+), \qquad (\tfrac{1}{2}^-,\ 1^-) \text{ and } (\tfrac{3}{2}^-,\ 1^-),$$

can be ruled out. Recently Yang and Lee [16]) have speculated that all particles whose strangeness is odd must exist in two states of opposite parity. In view of the mysterious degeneracy of the τ and θ, we had better to keep in mind the possible existence of the Λ_1^0 and Λ_2^0 which are degenerate with each other except for the space parity. For the decay probability of the $\Lambda \to \mathcal{N}+\pi$ the parity of Λ has also interesting effects [1]).

Assuming that both the Λ_1^0 and the Λ_2^0 have the interaction $\Lambda \to \mathcal{N}+\pi$, the simplest coupling types are as follows:

for $S(\Lambda_1^0) = \tfrac{1}{2}^+$;

$$g_1 \bar{\psi}_\mathrm{P} \gamma_5 \psi_{\Lambda_1} \varphi_{\pi^-}^* + g_1' \bar{\psi}_\mathrm{N} \gamma_5 \psi_{\Lambda_1} \varphi_{\pi^0}^* + \text{c.c.}, \tag{9}$$

$$g_2 \bar{\psi}_\mathrm{P} \gamma_5 \gamma_\mu \psi_{\Lambda_1} \partial_\mu \varphi_{\pi^-}^* + g_2' \bar{\psi}_\mathrm{N} \gamma_5 \gamma_\mu \psi_{\Lambda_1} \partial_\mu \varphi_{\pi^0}^* + \text{c.c.}; \tag{10}$$

for $S(\Lambda_2^0) = \tfrac{1}{2}^-$;

$$g_3 \bar{\psi}_\mathrm{P} \psi_{\Lambda_2} \varphi_{\pi^-}^* + g_3' \bar{\psi}_\mathrm{N} \psi_{\Lambda_2} \varphi_{\pi^0}^* + \text{c.c.}, \tag{11}$$

$$g_4 \bar{\psi}_\mathrm{P} \gamma_\mu \psi_{\Lambda_2} \partial_\mu \varphi_{\pi^-}^* + g_4' \bar{\psi}_\mathrm{N} \gamma_\mu \psi_{\Lambda_2} \partial_\mu \varphi_{\pi^0}^* + \text{c.c.}; \tag{12}$$

where

$$g_i = \sqrt{2}\,\alpha_i + \beta_i, \qquad g_i' = -\alpha_i + \sqrt{2}\,\beta_i, \qquad (i = 1, 2, 3, 4). \tag{13}$$

The subscript i of α and β corresponds to the choice of the coupling type g_i.

The values of g_i adjusted to the observed life-time of $\Lambda^0 \to \mathrm{P}+\pi^-$ (3×10^{-10} sec) are:

direct coupling:

$$g_1^2/4\pi \approx 5.2\times10^{-12}, \qquad g_3^2/4\pi \approx 1.4\times10^{-14}; \tag{14}$$

† We take $(S_\mathcal{N},\ S_\pi) = (\tfrac{1}{2}^+,\ 0^-)$.

gradient coupling:

$$(g_2^2/4\pi)m_\pi^2 \approx 2.4\times10^{-14},$$
$$(g_4^2/4\pi)m_\pi^2 \approx 8.5\times10^{-15}. \qquad (15)$$

Compare with the values of the coupling constant of $\pi \to \mu+\nu$ decay; $g^2/4\pi = 2.0\times10^{-15}$ for direct coupling and $(g^2/4\pi)m_\pi^2 = 3.6\times10^{-15}$ for gradient couplings. Except for the direct γ_5 type (g_1) interaction the value of g_i coincides for both decays within the factor 3. The best fit is given in the case g_4. This is interesting because Wentzel's argument also favours $(S_\Lambda, S_\theta)=(\frac{1}{2}^-, 0^+)$ and this derivative coupling is consistent with the rarity of the $\pi \to e+\nu$ and $K \to e+\nu$ modes. It must be noted that, even if we keep the universality of the coupling constant † (for instance, $g_1 = g_3$ or $g_2 = g_4$), covariance properties give different decay life-times for Λ_1^0 and Λ_2^0. That is, $\tau(\Lambda_1^0 \to P+\pi^-)/\tau(\Lambda_2^0 \to P+\pi^-) \approx 2.7\times10^{-3}$ for $g_1 = g_3$ and $\approx 3.5\times10^{-1}$ for $g_2 = g_4$. The branching ratio $R(\Lambda^0)$ (the determination of $R(\Lambda_1^0)$ and $R(\Lambda_2^0)$, if possible, is more desirable) will give information about the behaviour of $\Lambda^0 \to \mathcal{N}+\pi$ in isobaric spin space ††.

It will be shown in the next section that further restrictions are imposed by the consideration of the θ-decay.

5. θ^0-decay and τ^+-decay in our Model

5.1. θ^0-DECAY

In this paper, we are taking the model in which the θ-decay is caused by the Λ-decay interaction, which is assumed to be the basic weak interaction of the baryon-meson system. Before the discovery of the Λ^0, the $\theta \to 2\pi$ and $\tau \to 3\pi$ were discussed as the processes caused by the intermediation of nucleon loops [16]). There the selection rules due to the charge conjugation and the charge symmetry were quite powerful. However, because of the appearance of weak interactions which are not charge-independent and of particles which have non-vanishing strangeness quantum numbers, the above selection rules cannot be valid. As stated in § 3, the only way to obtain selection rules is to use the transformation property of $\theta \to \pi+\pi$ in isobaric spin space. (See, however, also §6.2).

The $\Lambda^0 \to \mathcal{N}+\pi$ interaction is so weak that it cannot appear more than once. Other virtual interactions must be strong G-int. (e-int. corrections are neglected at the moment) which are charge-independent. Thus, in our model, the transformation properties of the $\theta \to \pi+\pi$ in isobaric spin-space are completely determined by those of the $\Lambda^0 \to \mathcal{N}+\pi$. For instance, if we take for the $\Lambda \to \mathcal{N}+\pi$

† From the standpoint that f-int. is primary, this means, for example, that we take $f_1(\overline{\mathcal{N}}\gamma_5\Lambda_1)(\overline{\mathcal{N}}\gamma_5\overline{\mathcal{N}})$ and $f_2(\overline{\mathcal{N}}\Lambda_2)(\overline{\mathcal{N}}\gamma_5\mathcal{N})$ with $f_1 = f_2$ etc.

†† Steinberger's [6]) results on $R(\Lambda^0)$ lend little support to $\Delta I = \pm\frac{1}{2}$ for $\Lambda^0 \to \mathcal{N}+\pi$, if we assume that only one Λ^0 exists which decays by the mode $\Lambda^0 \to \mathcal{N}+\pi$.

$$\Delta I = \pm\tfrac{1}{2}, \qquad \beta_i = 0. \tag{i}$$

all the weak baryon-meson interactions transform as the component of a spherical tensor of rank $\frac{1}{2}$, as long as we neglect the corrections due to e-int. So, as discussed in § 3,2, if the θ-meson has even parity, we can forbid $\theta^+ \to \pi^+ + \pi^0$ strictly by the above requirement (i).

For the odd-parity θ-meson, a mixture of $\Delta I = \pm\frac{1}{2}$ and $\Delta I = \pm\frac{3}{2}$ is required. That is

$$\Delta I = \pm\tfrac{1}{2},\ \Delta I = \pm\tfrac{3}{2} \text{ with } \beta_i = 2\sqrt{2}\,\alpha_i. \tag{ii}$$

Both (i) and (ii) give the branching ratio

$$R(\theta^0) = W(\theta^0 \to \pi^0 + \pi^0)/W(\theta^0 \to \pi^+ + \pi^-) = \tfrac{1}{2}.$$

In order to get a crude orientation concerning the quantitative results following from our standpoints, we evaluate the transition probabilities of the θ^0-decay. The θ^0 is assumed to be 0^+. For that part of the $\Lambda^0 \to \mathcal{N} + \pi$ interaction which gives the predominant contribution to the θ-decays, we must impose $\Delta I = \pm\frac{1}{2}$. For the covariance properties of the $\Lambda \to \mathcal{N} + \pi$ we choose the derivative coupling, g_2 and g_4, of § 4. Except for the fact that they are preferable from the viewpoint of the universality of the coupling constant (see § 4) and that they give convergent results for $\tau \to \pi + \pi + \pi$ there are so far no definite reasons for their choice. The strong G-int., the ps pion-nucleon and the possible $\Lambda - \mathcal{N} - \theta$ interactions have the following forms:

$\mathcal{N} \rightleftarrows \mathcal{N} + \pi$:

$$G\bar{\psi}_{\mathcal{N}} \tau \gamma_5 \psi_{\mathcal{N}} \varphi_\pi \qquad \psi_{\mathcal{N}} = \begin{pmatrix} \psi_P \\ \psi_N \end{pmatrix} \tag{16}$$

$$= G[\sqrt{2}\,\bar{\psi}_N \gamma_5 \psi_P \varphi^*_{\pi^+} + \sqrt{2}\,\bar{\psi}_P \gamma_5 \psi_N \varphi^*_{\pi^-} + \bar{\psi}_P \gamma_5 \psi_P \varphi^*_{\pi^0} - \bar{\psi}_N \gamma_5 \psi_N \varphi^*_{\pi^0}];$$

$\mathcal{N} \rightleftarrows \Lambda_1 + \theta, \qquad S(\Lambda_1) = \frac{1}{2}^+$:

$$F_1 \bar{\psi}_{\Lambda_1} \varphi^*_\theta \psi_{\mathcal{N}} + \text{c.c.} \qquad \varphi_\theta = \begin{pmatrix} \varphi_{\theta^+} \\ \varphi_{\theta^0} \end{pmatrix} \tag{17}$$

$$= F_1[\bar{\psi}_{\Lambda_1} \psi_P \varphi^*_{\theta^+} + \bar{\psi}_{\Lambda_1} \psi_N \varphi^*_{\theta^0}] + \text{c.c.};$$

$\mathcal{N} \rightleftarrows \Lambda_2 + \theta, \qquad S(\Lambda_2) = \frac{1}{2}^-$:

$$F_2 \bar{\psi}_{\Lambda_2} \gamma_5 \varphi^*_\theta \psi_{\mathcal{N}} + \text{c.c.} \tag{18}$$

$$= F_2[\bar{\psi}_{\Lambda_2} \gamma_5 \psi_P \varphi^*_{\theta^+} + \bar{\psi}_{\Lambda_2} \gamma_5 \psi_N \varphi^*_{\theta^0}] + \text{c.c.}.$$

Here we only take direct couplings.

The lowest order diagram for the $\theta^0 \to \pi^+ + \pi^-$ is shown in fig. 1. For the $\theta^+ \to \pi^+ + \pi^0$ there are two diagrams, figs. 2 and 3. These two diagrams interfere completely destructively with each other for the part of $\Lambda^0 \to \mathcal{N} + \pi$ interaction with $\Delta I = \pm\frac{1}{2}$. For the θ^0-decay this selection rule fails. The matrix-element of fig. 1 is given, for example, by

$$\mathfrak{M} = (\sqrt{2}\,G)F_2 g_4 k_{2\mu} \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p$$

$$\mathrm{tr}\left[\frac{1}{(\mathbf{p}\gamma)+(\mathbf{k}_1\gamma)-M_\mathrm{P}}\gamma_5\frac{1}{(\mathbf{P}\gamma)-M_\mathrm{N}}\gamma_\mu\frac{1}{(\mathbf{P}\gamma)-(\mathbf{k}_2\gamma)-M_\Lambda}\gamma_5\right]$$

where k_1, k_2 and p are the energy-momentum four-vectors of the outgoing π^-, π^+ and the virtual proton respectively. To the order considered there are selection rules due to the covariance conditions in the ordinary space. The

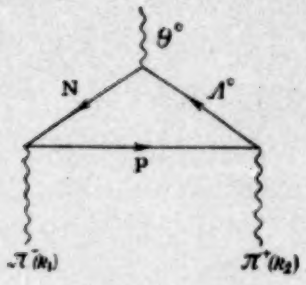


Fig. 1.

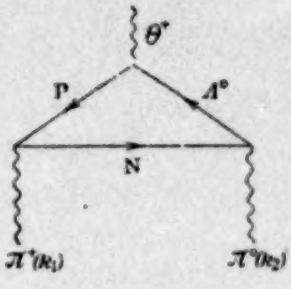
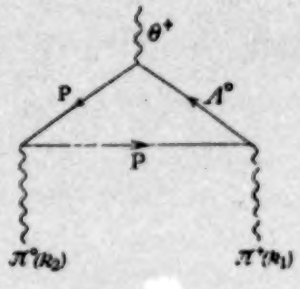


Fig. 2.



combinations GF_1g_1, GF_1g_2, GF_2g_3 and GF_2g_4 are allowed and others are forbidden.

Unfortunately, the matrix elements are logarithmically divergent. As there is no good remedy for it, we introduce a Feynmann convergence factor $-K^2/(p^2+K^2)$. The matrix-elements are complicated functions of M_Λ M_P, m_π and (k_1k_2). However, to a good approximation we can expand the results in powers of m_π^2/λ^2 and $(k_1k_2)/\lambda^2$ where λ is a quantity which is always larger than M_P. The mass-difference between M_Λ and M_P is included. Keeping only the lowest order term in the above expansions, the computation yields

$$\frac{1}{\tau} = \frac{G^2}{4\pi}\frac{F^2}{4\pi}\cdot\frac{g^2}{4\pi}m_\pi{}^2\cdot\frac{1}{4\pi^2}M_\mathrm{P}\left(1-4\frac{m_\pi{}^2}{m_\theta{}^2}\right)^{\frac{1}{2}}P$$

where P is a dimensionless quantity which depends on the cut-off momentum K. The fact that only half of the θ^0 can decay by the $\theta\to\pi+\pi$ mode is taken into account.

Let us take for G the value $G^2/4\pi \approx 15$ which is given by the present pion theory and for g_i the values given by (15). From the $\tau(\theta^0\to\pi^++\pi^-)$ $\approx 1.7\times10^{-10}$ sec we can evaluate the values of F_i which are shown in Table II for various cut-off K. The accurate value of F is not yet known. The values of F_2 (γ_5-type) listed are smaller than that of G for pion-nucleon interaction. However, we must note the s-wave damping in the ps theory. It is known that this theory taken in lowest order consistently overestimates the probability of processes involving virtual nucleon-antinucleon pair creation. For instance, the calculated life-time of $\pi^0\to 2\gamma$ on ps theory

is shorter than the experimental value by a factor 10^2. So if we expect the same effects also in the $\theta \to \pi+\pi$ case, the value of F_1 might be as large as that of G.

TABLE II

Case	K	$K = 2M_P$	$K = 5M_P$	$K = 10M_P$
$F_1 g_2$	P	1.75	1.25×10	3.19×10
$\frac{g_2^2}{4\pi} m_\pi^2$	$\frac{F_1^2}{4\pi}$	3.1×10^{-1}	4.4×10^{-2}	1.7×10^{-2}
$= 2.4\times10^{-14}$	$R=\frac{W(\theta^+\to\pi^++\pi^0)}{W(\theta^0\to\pi^++\pi^-)}$	1.23×10^{-4}	7.7×10^{-5}	6.9×10^{-5}
$F_2 g_4$	P	3.85	2.00×10	4.50×10
$\frac{g_4^2}{4\pi} m_\pi^2$	$\frac{F_2^2}{4\pi}$	4.0×10^{-1}	7.7×10^{-2}	3.4×10^{-2}
$= 8.5\times10^{-13}$	$R=\frac{W(\theta^+\to\pi^++\pi^0)}{W(\theta^0\to\pi^++\pi^-)}$	4.8×10^{-5}	4.3×10^{-5}	4.5×10^{-5}

5.2. τ-MESON DECAYS

The possible $\Lambda-\mathcal{N}-\tau$ interactions, taking the τ-meson as 0^-, are given by changing Θ into τ and exchanging Λ_1 and Λ_2 in (17) and (18):

$\mathcal{N} \rightleftarrows \Lambda_2+\tau$:

$$F_1'\overline{\Psi}_{\Lambda_2}\boldsymbol{\varphi}_\tau^*\psi_{\mathcal{N}}+\text{c.c.} \qquad \boldsymbol{\varphi}_\tau=\begin{pmatrix}\varphi_{\tau^+}\\ \varphi_{\tau^0}\end{pmatrix} \tag{19}$$

$$= F_1'[\bar\psi_{\Lambda_2}\psi_P\varphi_{\tau^+}^*+\bar\psi_{\Lambda_2}\psi_N\varphi_{\tau^0}^*]+\text{c.c.};$$

$\mathcal{N} \rightleftarrows \Lambda_1+\tau$:

$$F_2'\overline{\Psi}_{\Lambda_1}\gamma_5\boldsymbol{\varphi}_\tau\psi_{\mathcal{N}}+\text{c.c.} \tag{20}$$

$$= F_2'[\bar\psi_{\Lambda_1}\gamma_5\psi_P\varphi_{\tau^+}^*+\bar\psi_{\Lambda_1}\gamma_5\psi_N\varphi_{\tau^0}^*]+\text{c.c.}$$

In Yang and Lee's theory[17]), $F_1 = F_1'$ and $F_2 = F_2'$ are required †, from which results that θ and τ are always produced with equal abundance. The lowest order diagrams are given by fig. 4. (There is another graph which is obtained by exchanging k_1 and k_3.) Our choice of the gradient coupling for the $\Lambda \to \mathcal{N}+\pi$ interaction (g_2 and g_4) make the results convergent. The matrix elements are complicated functions of M_Λ, $M_{\mathcal{N}}$, m_π and (k_1k_2), (k_1k_3) etc. We take the same approximations as in the case of $\theta^0 \to \pi^+ + \pi^-$, which essentially amount to neglect quantities of the order $(m_\pi/M_P)^2$. The mass difference between M_Λ and $M_{\mathcal{N}}$ is included. The results are

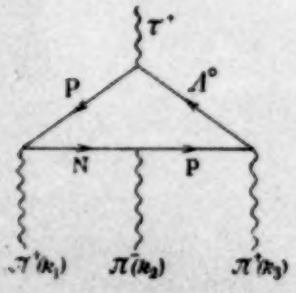


Fig. 4.

† It does not necessarily require that both coupling types must exist. See § 6.3.

$$\frac{1}{\tau}=\left(\frac{G^2}{4\pi}\right)^2\cdot\frac{F'^2}{4\pi}\cdot\frac{g^2}{4\pi}m_\pi{}^2\cdot\frac{1}{2^5}\frac{1}{\pi^3}m_\tau\left(\frac{m_\pi}{M_{\rm p}}\right)^2\left(\frac{m_\pi}{m_\tau}\right)^2 s \qquad (21)$$

where s is a dimensionless quantity which must be calculated. Assuming that $G^2/4\pi\approx F_i'^2/4\pi\approx 15$ $(i=1,2)$ and taking the values of g_i $(i=2,4)$ given by (15), we get the life-time τ of $\tau^+\to\pi^++\pi^++\pi^-$ †. The results for both s and τ are listed in table III.

TABLE III

Case	s	$\tau(\tau^+\to\pi^++\pi^++\pi^-)$
F_2' g_4	2.22	1.2×10^{-8} sec
F_1' g_2	0.906	8.2×10^{-8} sec

For the sake of comparison, we add the following results for other choices of coupling types. They can be recalculated from the results of ref. [16]), if we tolerate the approximation $M_\Lambda=M_{\rm p}$. We only take the convergent cases and assume that $F\approx G$.

(i) $\bar{\Lambda}_1\mathcal{N}\tau$ (ps-coupling) $\dfrac{F_2'^2}{4\pi}\approx 15,$

$\bar{\mathcal{N}}\mathcal{N}\pi$ (pv-coupling) $\dfrac{G^2}{4\pi}m_\pi{}^2\approx 0.1,$

$\bar{\Lambda}_1\mathcal{N}\pi$ (pv-coupling) $\dfrac{g_2^2}{4\pi}m_\pi{}^2\approx 2.4\times10^{-14}$ from (15),

$\tau(\tau^+\to\pi^++\pi^++\pi^-)\approx 5.0\times10^{-5}$ sec.;

(ii) $\bar{\Lambda}_1\mathcal{N}\tau$ (pv-coupling) $\dfrac{F_2'^2}{4\pi}m_\pi{}^2\approx 0.1,$

$\bar{\mathcal{N}}\mathcal{N}\pi$ (ps-coupling) $\dfrac{G^2}{4\pi}\approx 15,$

$\bar{\Lambda}_1\mathcal{N}\pi$ (ps-coupling) $\dfrac{g_1^2}{4\pi}\approx 5.2\times10^{-12}$ from (14),

$\tau(\tau^+\to\pi^++\pi^++\pi^-)\approx 1.6\times10^{-8}$ sec.

It is seen that, to the lowest order considered, derivative type strong $\bar{\mathcal{N}}\mathcal{N}\pi$ coupling with $(G^2/4\pi)m_\pi{}^2\approx 0.1$ gives too small probabilities. In our simplified model, according to which among baryons only the Λ^0 is elementary, the contributions of other hyperons (Σ, Ξ etc.) can appear only in higher order processes. However, if the strength of the coupling

† If both θ and τ have the $K\to\mu+\nu$ mode, the greater abundance of this mode among K-meson decays is consistent with taking the partial lifetime of $\tau^+\to\pi^++\pi^++\pi^-$ as being of the order of 10^{-7} sec.

constants of, say, $\bar{\Sigma}\mathcal{N}\mathrm{K}$ and $\bar{\Sigma}\Lambda\pi$ interactions turns out to be of the same order as that of $\bar{\Lambda}\mathcal{N}\mathrm{K}$ and $\bar{\mathcal{N}}\mathcal{N}\pi$, the contributions of such intermediations as

$$\theta \to \mathcal{N}+\bar{\Sigma} \to \mathcal{N}+(\bar{\Lambda}+\pi) \to \pi'+\pi$$
$$\tau \to \mathcal{N}+\bar{\Sigma} \to \mathcal{N}+(\bar{\Lambda}+\pi) \to (\mathcal{N}+\pi')+(\bar{\Lambda}+\pi) \to \pi''+\pi'+\pi$$

may not be so small compared with those which we calculated in this paper. At the moment we cannot go into detail because of the lack of knowledge about these interactions. Therefore, our results must be taken as giving an orientation only.

5.3. FREQUENCY RATIO OF $\tau^+ \to \pi^+ + \pi^0 + \pi^0$ AND $\tau^+ \to \pi^+ + \pi^+ + \pi^-$

Next, let us investigate the frequency ratio

$$R(\tau^+) = W(\tau^+ \to \pi^+ + \pi^0 + \pi^0)/W(\tau^+ \to \pi^+ + \pi^+ + \pi^-).$$

For the $\tau^+ \to \pi^+ + \pi^0 + \pi^0$, to the lowest order, there are six independent diagrams corresponding to the possibility that the outgoing π^+ can be emitted from three different vertices. This time, the interactions with the coupling constants g_i' also contribute. The life-time of $\tau^+ \to \pi^+ + \pi^0 + \pi^0$ is given by

$$W(\tau^+ \to \pi^+ + \pi^0 + \pi^0) = \left(\frac{G^2}{4\pi}\right)^2 \cdot \frac{F'^2}{4\pi} \cdot \frac{g^2}{4\pi} m_\pi{}^2 \cdot \frac{1}{2^5} \frac{1}{\pi^3} m_\tau \left(\frac{m_\pi}{M_p}\right)^2$$
$$\cdot \left(\frac{m_\pi}{m_\tau}\right)^2 \cdot \frac{1}{16}\left[p + \left(\frac{g'}{g}\right)q + \left(\frac{g'}{g}\right)^2 r\right]$$

and so $R(\tau^+)$ becomes, from (21),

$$R(\tau^+) = \frac{1}{16}\frac{1}{s}\left\{p + \left(\frac{g'}{g}\right)q + \left(\frac{g'}{g}\right)^2 r\right\} \tag{22}$$

where, for the case $F_1' g_2$,

$$p/s \approx 4.0, \qquad q/s \approx -0.038, \qquad r/s \approx 0.030,$$

and for the case $F_2' g_4$,

$$p/s \approx 4.0, \qquad q/s \approx +0.11, \qquad r/s \approx 0.24.$$

Note that we neglect the mass difference between π^+ and π^0 which will slightly increase the value of $R(\tau^+)$.

From (22) the reality condition of g'/g gives for both cases

$$R(\tau^+) \geqq 0.25.$$

As in both cases $|p/s| \gg |q/s|$ and $|p/s| \gg |r/s|$ we always get, for $|g'/g| \approx 1$ (Note that $g'/g = -1/\sqrt{2}$ for $\Delta I = \pm\frac{1}{2}$ and $g'/g = \sqrt{2}$ for $\Delta I = \pm 3/2$),

$$R(\tau^+) \approx \frac{1}{16}\frac{p}{s} \approx \frac{1}{4}. \tag{23}$$

This means that the value $R(\tau^+) \approx \frac{1}{4}$ which seems to be indicated by the present experiments is not sensitive to the choice of $\Delta I = \pm\frac{1}{2}$ or $\Delta I = \pm\frac{3}{2}$ for $\Lambda \to \mathcal{N}+\pi$. So from $R(\tau^+)$ we cannot extract any conclusion about the choice between the above two possibilities. We use this fact in the discussion of § 6.3. Eq. (23) implies that in the case of a ps τ-meson the final state with $I_f = 1$ is predominant.

6. Possible Interactions Responsible for the Occurrence of the $\theta^+ \to \pi^+ + \pi^0$

It is known that the $\theta^+ \to \pi^+ + \pi^0$ is strictly forbidden, by the rule $\Delta I = \pm\frac{1}{2}$, for even space parity θ. In order to make the $\theta^+ \to \pi^+ + \pi^0$ allowed, we must invoke a smaller component which transforms in charge space as a spherical tensor of the order $\frac{3}{2}$. Here we briefly discuss the various possibilities.

6.1. ELECTROMAGNETIC INTERACTION

If we want to choose the simplest transformation property of the basic weak interaction, i.e. a spherical tensor of order $\frac{1}{2}$, the only way is to expect that electromagnetic effects are fully responsible for the required inclusion of a component of order $\frac{3}{2}$. At first, this electromagnetic correction of order e^4 seems too small. However, this first impression may be deceptive. For instance, the mass-difference between π^+ and π^0 is larger (about $9m_e$) compared to fine structure constant orders of magnitude. Here we investigate the results of the insertion of the mass-difference of the particles under consideration which amounts to taking partially into account the full effects of the higher order corrections (self-energy type) due to electromagnetic interaction.

First, it is instructive to note the effect on the basic $\Lambda^0 \to \mathcal{N}+\pi$ interaction which is required to satisfy the property $\Delta I = \pm\frac{1}{2}$. The ratio of the square of the amplitudes of $\Lambda^0 \to \mathrm{P}+\pi^-$ and $\Lambda^0 \to \mathrm{N}+\pi^0$,

$$R = M(\Lambda^0 \to \mathrm{P}+\pi^-)/M(\Lambda^0 \to \mathrm{N}+\pi^0),$$

is given for each case by

$$g_1 : \frac{1}{1.076} \times 2, \qquad g_3 : \frac{1}{1.001} \times 2,$$

$$g_2 : \frac{1}{1.077} \times 2, \qquad g_4 : 1.010 \times 2;$$

R would be $= 2$ if we did not include the mass-differences between $\pi^\pm$ and π^0 and between proton and neutron. Except for the case g_3, the effects are fairly large compared with the expectation of e^4-effects, though this may come from the smallness of the Q-value of this decay. Thus, if this

effect is exactly reflected in the θ^+-decay which is, in our model, due to the Λ-decay, we may explain the strange frequency of θ^+-decay by the electromagnetic effect. However, the results of the actual calculations for $R = W(\theta^+ \to \pi^+ + \pi^0)/W(\theta^0 \to \pi^+ + \pi^-)$ listed in table II fail to support the above arguments, at least for the coupling types adopted here.

In the approximation adopted in § 5.1, the matrix elements $\mathfrak{M}_2$ and $\mathfrak{M}_3$ of figs. 2 and 3 are of the form

$$\mathfrak{M}_2 = (k_1 k_2)A + k_2^2 B, \qquad k_2^2 = m_{\pi^0}^2;$$
$$\mathfrak{M}_3 = (k_1 k_2)A + k_1^2 B, \qquad k_1^2 = m_{\pi^+}^2;$$

A and B are scalar functions of M_Λ, $M_\mathcal{N}$, k_1^2, k_2^2 and $(k_1 k_2)$. In our approximation the leading terms of A and B do not depend on k_1^2, k_2^2 and $(k_1 k_2)$. The frequency ratio is given by

$$R \equiv \frac{W(\theta^+ \to \pi^+ + \pi^0)}{W(\theta^0 \to \pi^+ + \pi^-)} \approx \frac{(\mathfrak{M}_2 - \mathfrak{M}_3)^2}{\mathfrak{M}_2^2} = \frac{(k_1^2 - k_2^2)^2 B^2}{\{(k_1 k_2)A + k_2^2 B\}^2}.$$

If R is proportional to $\{(k_1^2 - k_2^2)/k_2^2\}^2 = (\Delta m_\pi^2/m_\pi^2)^2 \approx 10^{-2}$, we get the desired results. However, as $(k_1 k_2) \approx 5.3\, k_1^2$ and it turns out that $A \approx B$ for our choice of coupling types and of cut-off momentum † K, the results are not satisfactory. The values given in table III include not only the mass-difference of the pions, but also that of nucleons. If higher-order corrections, due to strong baryon-meson interaction and also vertex type corrections due to e-int. which are not included in the above discussion, do not change the relation between A and B, the ratio $R \approx 10^{-4}$ may be the general feature. It might, however, be premature to take this as a definite indication for the necessity of the existence of basic weak interactions which have the transformation property of rank $\frac{3}{2}$ in isobaric spin space.

6.2. FINAL STATE INTERACTIONS

In our discussion, we do not so far take into account the final state interaction between outgoing particles of K-decays and Λ-decays. Okubo and Marshak [19]) suggest the possibility that the difference between the θ^0 and θ^+ life-times could be due to an interaction between π^+ and π^- in the $I = 0$ state (which is not available for the pion state in $\theta^+ \to \pi^+ + \pi^0$) provided the range of interaction is of order $\hbar/m_\pi c$ and not $\hbar/M_\mathcal{N} c$. This standpoint, of course, requires that the weak interaction must have an appreciable component of the tensor of order $\frac{3}{2}$. On the other hand, the following standpoint can also be possible, which sticks to the simplest transformation property of the weak interaction $\Delta I = \pm\frac{1}{2}$. As in § 6.1, we consider that electromagnetic interaction is the only source which can

† In table III we take $K \geqq 2M_\mathrm{P}$. In connection with s-wave damping it may be interesting to investigate the cut-off $K \leqq M_\mathrm{P}$. In fact, the life-time of $\pi^0 \to 2\gamma$ is consistent with the smaller cut-off [18]).

remove the strict exclusion of $\theta^+ \to \pi^+ + \pi^0$. If we assume, contrary to Okubo and Marshak, strong interaction between π^+ and π^0 in the $I = 1$ state, it will also increase the probability of the $\theta^+ \to \pi^+ + \pi^0$ obtained in § 6.1 ($R \approx 10^{-4}$) up to the experimental value. A slightly more complicated proposal was made by Iso [14]).

6.3. INCLUSION OF $\Delta I = \pm 3/2$ FOR WEAK INTERACTION

If both § 6.1 and § 6.2 do not work we may be forced to introduce a component of order $\frac{3}{2}$ ($\Delta I = \pm\frac{3}{2}$) in the basic weak interactions. Unless Okubo and Marshak's idea does work, the main parts of the $\theta \to \pi + \pi$ must behave as a tensor of order $\frac{1}{2}$ in order to make the $\theta^+ \to \pi^+ + \pi^0$ first forbidden. As shown in § 5.3, for the τ^+-decays $\Delta I = \pm\frac{1}{2}$ and $\Delta I = \pm\frac{3}{2}$ give no difference for the branching ratio $R(\tau^+)$. So we can present one possible model which keeps the universality of the coupling constant g_i. For the strong $\Lambda - \mathcal{N} - \mathrm{K}$ interactions, we have two coupling types (F_1, F_1') and (F_2, F_2') according to the choice of the combinations of Λ_1, Λ_2 and θ, τ. Let us, however, require that only one of them, say, (F_2, F_2') is allowed to exist. For the $\Lambda \to \mathcal{N} + \pi$, there are also two types, γ_5 type (g_1, g_2) and non-γ_5 type (g_3, g_4), according to whether Λ is Λ_1 or Λ_2.

We impose the condition $\Delta I = \pm\frac{1}{2}$ for one of them and $\Delta I = \pm\frac{3}{2}$ for the other †. The choice of the above assignments is taken so as to make the lowest order processes $\theta^+ \to \pi^+ + \pi^0$ forbidden for the part $\Delta I = \pm\frac{3}{2}$ according to the selection rules due to covariance requirements discussed in § 5.1. For instance, corresponding to the above choice (F_2, F_2') one has $\Delta I = \pm\frac{1}{2}$ for (g_3, g_4) and $\Delta I = \pm\frac{3}{2}$ for (g_1, g_2). Then $\theta^+ \to \pi^+ + \pi^0$ can occur by the higher order processes which contain the weak vertex $\Delta I = \pm\frac{3}{2}$. As these higher order corrections need not include e-int. it will not be so difficult to get the required frequency of $\theta^+ \to \pi^+ + \pi^0$ as in the case of electromagnetic correction discussed in § 5.1. Note also that the lowest order τ-decay is given by $F_2' g_4$ in table III or by $F_2' g_3$.

Note added in proof: If space parity is not conserved in weak interactions, as suggested recently by Yang and Lee[21]), the $\mathrm{K}_{2\pi}$-decay and the $\mathrm{K}_{3\pi}$-decay would be interpreted as different decay-modes of a unique K-meson and also the introduction of two Λ^0's, Λ_1 and Λ_2, would not be necessary. However, the selection rules for the $\mathrm{K}_{2\pi}$-decay used in our discussion need not be changed. If we have only one Λ^0, say Λ_2, then our Λ-decay interactions of the form g_1 and g_2 become parity non-conserving interactions. Our proposal in 6.3 can then be expressed in the following way. Take, e.g., the spin and parity assignment $(S_A, S_\mathrm{K}) = (\frac{1}{2}^-, 0^+)$. Then the strong Λ—$\mathcal{N}$—K interaction is of the γ_5-type given by (18). We assume $\Delta I = +\frac{1}{2}$ for parity-conserving interactions g_3 or g_4 and impose $\Delta I = \pm^3/_2$ on parity non-conserving ones g_1 or g_2.

7. Concluding Remarks

In this paper we have shown that the various K-meson decays into pions can possibly be explained by taking the $\Lambda \to \mathcal{N} + \pi$ interaction as the main

† This can be taken as a revision of the discussion of § 2.1 in I taking into account the possible existence of Λ_1^0 and Λ_2^0.

link of the weak interactions. Of course, our analysis meets with a difficulty due to the common feature of the present field theory, that is, the problem of divergency and the lack of a good method to treat strong interactions in a relativistically covariant way. However, leaving these problems to the future, we feel that we have rather good orientation as to the internal consistency of our standpoint. We close our discussion by adding the following remarks.

Here we assumed that the $\Lambda \to \mathcal{N} + \pi$ interaction is the source of weak interactions. In I, we comprise every hyperon decay interaction into the category of g-int. In this case we can also follow, in principle, the same line as our present analysis replacing the role of $\Lambda \to \mathcal{N} + \pi$ interaction by the whole baryon-meson g-int.

The composite particle idea of Sakata and Markov mentioned in § 1 is quite interesting if it is successful quantitatively. In this model θ is regarded as the ground state of the nucleon-anti-Λ^0 system. If we take Yang and Lee theory [17]), we may regard, for instance, θ and τ as ($\mathcal{N}$, $\overline{\Lambda}_2^0$) and (N, $\overline{\Lambda}_1^0$) respectively. Then θ-decays and τ-decays can be inter-

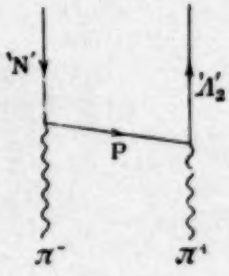


Fig. 5.

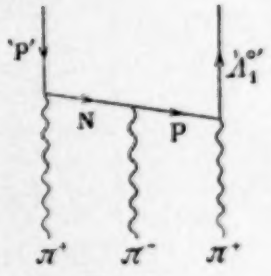


Fig. 6.

preted as due to '$\overline{\Lambda}$'-decays within the bound states. $\Theta^0 \to \pi^+ + \pi^-$ and $\tau^+ \to \pi^+ + \pi^+ + \pi^-$ are shown in figs. 5 and 6, respectively, (' ' indicates a bound state.) In this theory, our standpoint of taking the $\Lambda \to \mathcal{N} + \pi$ as the main source of weak baryon-meson interaction is quite natural. As discussed in I, we can replace this interaction by the f-int., say, $\Lambda \gtrless \mathcal{N} + \mathcal{N} + \mathcal{N}$. Thus we can possibly explain every weak reaction by the universal Fermi interactions which exist between elementary particles. Thus our phenomenological universal interactions may be interpreted as the result of the existence of the fundamental interactions between truly elementary entities such as nucleon, Λ^0, lepton and photon.

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SUM RULE FOR OPTICAL NUCLEAR REACTIONS

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Abstract: The direct or optical component of nuclear reactions is computed essentially by a first-order perturbation procedure. It should thus be subject to simple sum rules. One such rule is applied to elastic and inelastic α-particle scattering on Ne^{20}, where it is in qualitative accord with the observed smoothing out of fluctuations in the summed angular distributions.

Optical models have been useful to interpret nuclear elastic scattering of heavy particles. For inelastic scattering the same approach predicts a "direct" component in addition to that associated with compound nucleus formation. The optical inelastic component is calculated to first order by perturbation theory, treating the particle-nucleus interaction as weak. Such a calculation admits various sum rules, which can be written simply under certain approximations. The present note concerns one such approximate sum rule. It cannot have complete validity, but its qualitative implication should be correct: although the differential cross sections for elastic and inelastic scattering may individually show considerable fluctuations as a function of angle, the sum of these cross sections will tend to be a smooth, forward-peaked function of angle. That is, the peaks of one $d\sigma(\theta)$ tend to compensate the valleys of another. An appropriate set of experimental data is compared with this rule and found to be in moderate agreement.

At high incident energies the scattering to low-lying residual levels should be predominantly "direct", since the compound nucleus component suffers strong competition from more highly inelastic and other reactions. With neglect of exchange effects the direct or optical cross section is

$$d\sigma_{0f}(\theta) \approx \left(\frac{k_f}{k_0}\right) \left| \sum_{j=1}^{A} \int \Psi_f^*(r_1 \ldots r_A) e^{-i\Delta\mathbf{k}\cdot\mathbf{r}} V_j(r-r_j) \Psi_0(r_1 \ldots r_A) d\tau^A \, d\tau \right|^2 \qquad (1)$$

Here $\Delta\mathbf{k} = \mathbf{k}_f - \mathbf{k}_0$ for the scattered particle and V_j is the potential (including Coulomb interaction) between the incident particle and the jth target nucleon. Since the final nuclear states Ψ_f form a complete set, we can use closure on the sum over all f, provided that Δk is kept constant and the factor (k_f/k_0) is extracted; thus

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$$\begin{aligned} d\sigma(\Delta k) &\equiv \sum_f (k_0/k_f) d\sigma_{0f}(\Delta k) \equiv \sum_f d\sigma_f(\Delta k) \\ &\approx \sum_j \int e^{-i\Delta\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} V_j^*(r-r_j) V_j(r'-r_j) \varrho_j(r_j) d\tau_j\, d\tau\, d\tau' \end{aligned} \tag{2}$$

where we have written Ψ_0 as a product of individual-particle orbitals $\phi_j(r_j)$, and $\varrho_j = |\phi_j|^2$. For convenience in notation, the quantity $d\sigma_f$ is defined throughout as (k_0/k_f) times the actual differential cross section. If $V_j(r)$ is real, $V_j(k) = \int V_j(r)\, e^{-i\mathbf{k}\cdot\mathbf{r}} d\tau = V_j^*(-k)$, and (2) becomes

$$d\sigma(\Delta k) \approx \sum_j V_j(\Delta k) V_j(-\Delta k) \int \varrho_j d\tau_j \approx \sum_j |V_j(\Delta k)|^2. \tag{3}$$

The function $|V_j(\Delta k)|^2$ should vary rather smoothly with θ, since the nuclear part of $V_j(r)$ varies rapidly only in a narrow range of r, corresponding to a wide range of Δk. Because the $V_j(r)$ includes a Coulomb interaction, the $V_j(\Delta k)$ will show the usual Coulomb "spike" at forward angles, plus some interference structure at slightly larger Δk where nuclear and Coulomb parts of $V_j(\Delta k)$ are comparable. This interference structure should again be relatively smooth because it corresponds to scattering by a single proton rather than by a whole nucleus. Inclusion of the Coulomb potential in V_j means that (3) incorporates to first order some Coulomb interference in elastic scattering and also Coulomb excitation of residual levels, which is not distinguished experimentally from nuclear interaction.

The summed cross section (3) should be a smoother function of angle than the individual cross sections $d\sigma_{0f}$, whose form factors $f(\Delta k)$ correspond to relatively wide spatial regions on the order of the target nucleus volume. Of course no experimental sum can include an infinite number of final states, as required for the ideal limit (3). One can argue, however, that the convergence of the experimental sum may be fairly rapid. In eq. (1) write $\Psi_f = \sum_s P_s \Psi_f$, where the P_s are projection operators for various symmetries of the final state wave function. Then since $P_s P_{s'} = \delta_{ss'} P_s$, the sum rule can be written

$$d\sigma(\Delta k) \approx \sum_s \sum_j \int \Psi_0^* V_j^*(\Delta k) P_s V_j(\Delta k) \Psi_0 d\tau^A \tag{4}$$

which reduces to (3) upon performing the summation over symmetries. Since the V_j can effect transitions only of single nucleons, most of the terms in the $\sum_s$ will vanish; the only contributions come from final states with symmetries not much different from those in the initial state (which may contain more than one symmetry). These relatively symmetric final states will tend to cluster at low energies, so that a major fraction of the experimental sum may be contributed by the lowest lying residual states. Since each term in the summation $\sum_s$ is itself a relatively smooth function of Δk, the experimental $\sum_f d\sigma_f$ breaks down into a number of partial sum

rules on grouping the final states f into symmetry classes s. Thus one may say qualitatively that the degree to which two experimental $d\sigma_f(\Delta k)$ tend to compensate each other's fluctuations is a measure of the similarity of the symmetries of the corresponding Ψ_f. Of course all these considerations relate to the "direct" scattering component and apply to the experimental cross sections only if the compound nucleus component is relatively small. This is generally assured by the condition of high incident energy and relatively low energy loss but will be least true for large Δk, where the direct component becomes small. We thus expect that the greatest irregularity in the summed cross section will occur for the largest Δk values.

Equation (3) neglects multiple scattering; this is serious in principle but in practice should not invalidate the qualitative rule about smoothing of summed angular distributions. The multiple scattering gives rise to an optical potential inside the nucleus, so that the argument of V in equation (3) is no longer exactly the Δk of the scattering event. One has instead $\bar{V}(\Delta k) = \int V(k) f(k-\Delta k) d^3k$, an average of V around the value Δk, where f describes the mixing of k values by the optical potential. This averaged $\bar{V}(\Delta k)$ should be *a fortiori* a smooth function of Δk, however, preserving the tendency to compensation of peaks and valleys of individual angular distributions.

This qualitative conclusion would also hold for scatterings that are supposed to occur mainly near the nuclear surface [1, 2]). This would be crudely represented by the insertion in equation (2) of a "damping factor" $d_j(r_j)$ that is unity in the region of the nuclear surface and becomes small inside the nucleus. The presence of this factor will not affect the Δk dependence of equation (3).

For an experimental test of these remarks, we should like to compare individual $d\sigma_f(\Delta k)$ with the summed $d\sigma(\Delta k)$ and see the extent to which the fluctuations are smoothed out in the summation. Since high incident energies require cyclotron acceleration, individual levels can be resolved only for light nuclei.

Recent measurements have been made [3]) of angular distributions for the scattering of 20 MeV α-particles to the ground and 1st, 2nd, 3rd, and 4th excited states of Ne^{20}. Figures 1—3 show preliminary measurements on the elastic scattering distribution $d\sigma_0$ and the effects of adding successively the inelastic distributions $d\sigma_1$ and $(d\sigma_2+d\sigma_3+d\sigma_4)$. The smooth dotted curves are sketched in merely to provide reference lines for judging the degree of fluctuation in the solid curves. The most pronounced smoothing occurs in adding $d\sigma_1$ to $d\sigma_0$: these two distributions show a strong tendency to compensating peaks and valleys. The fluctuations at $\Delta k > 2\times 10^{13}$ cm^{-1} are not compensated and may represent compound nucleus effects. The use of the smoothing criterion to indicate levels with comparable symmetries

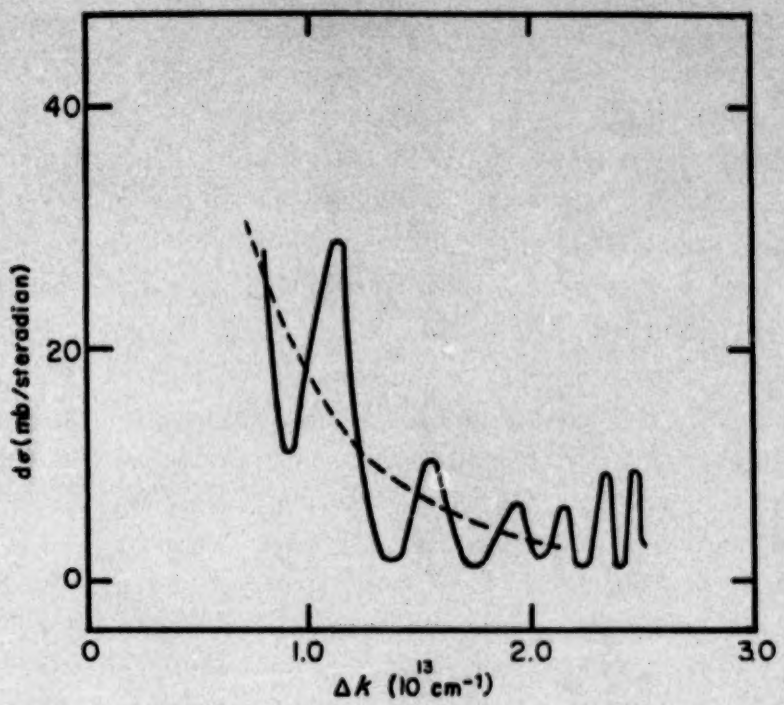

Fig. 1. Differential cross section $d\sigma_0$ for elastic scattering of 20 MeV α-particles on Ne^{20}; dotted line is for visual estimate of fluctuations only.

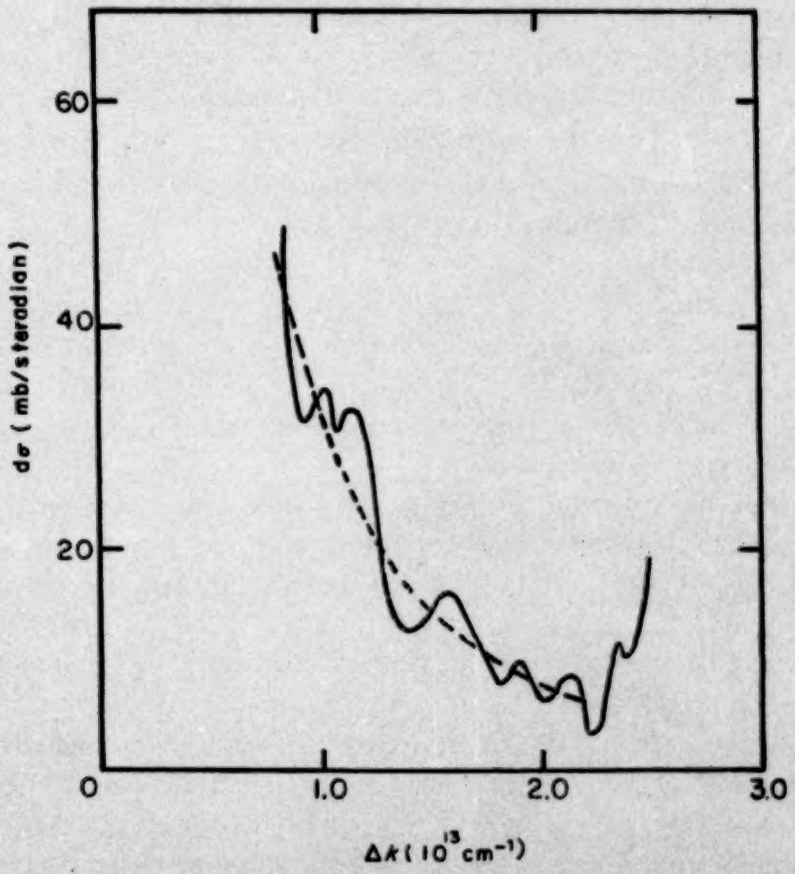

Fig. 2. Cross section sum $(d\sigma_0+d\sigma_1)$ for same case as fig. 1; the differential cross sections $d\sigma_l$ are (k_0/k_l) times the measured values.

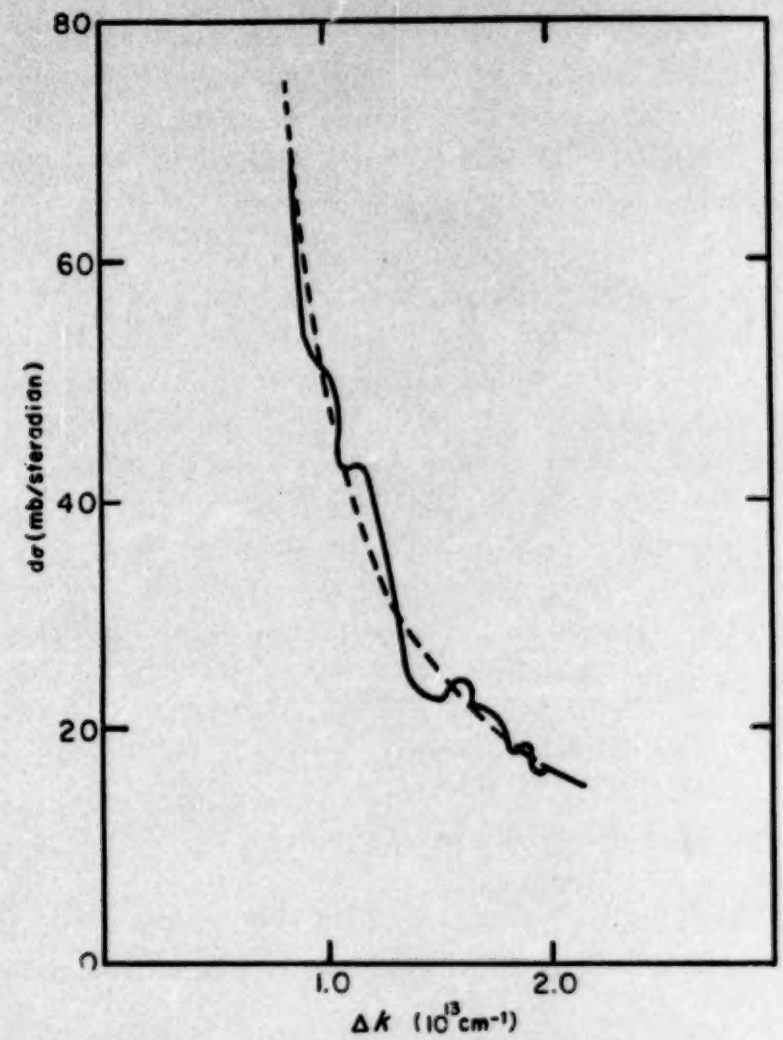

Fig. 3. Cross section sum $(d\sigma_0+d\sigma_1+d\sigma_2+d\sigma_3+d\sigma_4)$ for same case as fig. 1.

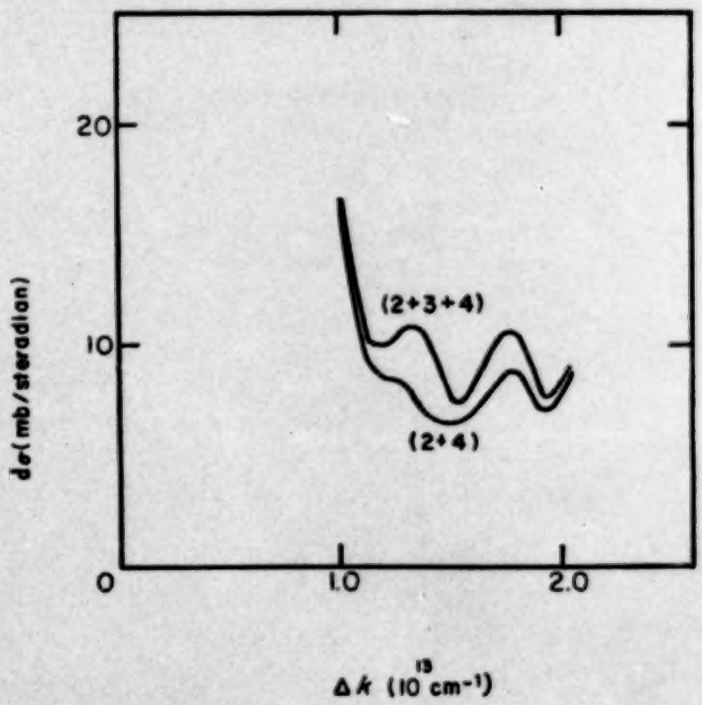

Fig. 4. Comparison of $(d\sigma_2+d\sigma_4)$ and $(d\sigma_2+d\sigma_3+d\sigma_4)$.

is illustrated in fig. 4: the curve of $(d\sigma_2+d\sigma_4)$ is smoother than either component; but the addition of $d\sigma_3$ again increases the fluctuations. Thus the 2nd and 4th excited states of Ne^{20} are probably more similar in structure to each other than to the 3rd excited state. † Unfortunately, the dotted curve in fig. 3 does not extend to low enough angles to reveal much information about $V_j(r)$ for α-particles; one can only say that its root-mean-square radius of action appears to be of order 2×10^{-13} cm. A similar qualitative discussion can be made of the (p, p′) scattering [4]) on O^{16}.

Of course these same considerations apply to the "direct" component of nuclear reactions other than inelastic scattering, such as (p, n), (n, α), and of course stripping and pickup reactions like (d, p) and (p, d). Here the reaction occurs mainly at the nuclear surface, where the nucleon density and corresponding optical potential V_0 are small. The curves of (d, p) stripping for various l_n values and constant ΔE are known [5]) to show compensation of peaks and valleys to a high degree. Equation (3) is a little different in that it sums over nucleon levels with different ΔE and includes some Coulomb effects; the nuclear levels are also characterized by total J and parity, not necessarily by specific l_n. In this case the $|V(\Delta k)|^2$ should correspond to a spatial width approximating that of the entire target nucleus, since deuteron stripping and pickup are supposed to occur mainly at the nuclear surface.

The authors wish to thank Mr. L. Seidlitz and Professor E. Bleuler for communication of preliminary measurements on Ne^{20}.

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† It should perhaps be remarked that "similarity in structure" does not imply "equality of J values" — in fact, rather the opposite. A given symmetry class includes a set of various J values, possibly with little repetition; among many levels with identical J values, several different symmetry classes are presumably represented.

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A POSSIBLE EXPLANATION OF THE ANOMALOUS SCATTERING OF μ-MESONS

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Abstract: The elastic Coulomb scattering cross-section of μ-mesons at large angles is affected by inelastic scattering of the incident μ-mesons when the energy transfer is $\geqq 150$ MeV. This effect is found to give results which are not inconsistent with the observations on the large angle scattering μ-mesons by lead nuclei if $\sigma_{\text{inel}} \approx 4$ mb/nucleus.

1. Introduction

The scattering of μ-mesons through large angles by atomic nuclei has been discussed by many authors (see e.g. the survey by Amaldi [1]), and the more recent paper by Lloyd and Wolfendale [2])). It is generally agreed that this scattering exceeds that which whould be expected from an electromagnetic interaction with an extended nucleus, although some authors would impose an upper limit on the angular range over which this effect may be observed for a given momentum [3]).

The most recent work on the subject [2]) indicates that the variation of anomalous scattering with atomic number, its variation with momentum and the lack of appreciable energy loss are all suggestive of some form of elastic interaction. In the following note we propose to interpret this anomalous scattering as a modification of the elastic Coulomb scattering due to inelastic processes such as π-meson production by the incident μ-mesons. This last process was discovered by George [4]) in analysing stars in photographic plates exposed underground.

2. Calculation

The effect on the elastic scattering may be estimated by introducing imaginary parts into the elastic-scattering phase shifts [5]). These imaginary parts $i\varkappa_j$ will also determine the corresponding cross-section for inelastic scattering which can then be compared with experiment.

It is convenient to use the representation of Yennie *et al.* [6]), who give, for the amplitude of the scattered wave at angle θ,

$$f(\theta) = \frac{1}{2ik} \Sigma \, e^{2i\eta_j}(j+\tfrac{1}{2}) \{P_{j-\frac{1}{2}}(\cos\theta) + P_{j+\frac{1}{2}}(\cos\theta)\}, \qquad (1)$$

where the P's are Legendre polynomials, k is the wave number of the incident particle and

$$\eta_j = \lambda_j + i\varkappa_j \qquad (\lambda_j \text{ and } \varkappa_j \text{ real}).$$

The differential cross-section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2(1+\tan^2 \tfrac{1}{2}\theta). \tag{2}$$

We will now assume that $\varkappa_j = \varkappa$ for all $j \leqq j_{\max}$ and zero thereafter, so that

$$f(\theta) = \frac{1}{2ik}\left(\mathrm{e}^{-2\varkappa}\sum_{j=\frac{1}{2}}^{j=j_{\max}} \mathrm{e}^{2i\lambda_j}(j+\tfrac{1}{2})\{P_{j-\frac{1}{2}}(\cos\theta)+P_{j+\frac{1}{2}}(\cos\theta)\} + \sum_{j=j_{\max}}^{\infty} \mathrm{e}^{2i\lambda_j}(j+\tfrac{1}{2})\{P_{j-\frac{1}{2}}(\cos\theta)+P_{j+\frac{1}{2}}(\cos\theta)\}\right) \tag{3}$$

$$= f'(\theta) - \frac{1}{2ik}(1-\mathrm{e}^{-2\varkappa})\sum_{j=\frac{1}{2}}^{j=j_{\max}} \mathrm{e}^{2i\lambda_j}(j+\tfrac{1}{2})\{P_{j-\frac{1}{2}}(\cos\theta)+P_{j+\frac{1}{2}}(\cos\theta)\},$$

where $f'(\theta)$ is the Coulomb scattered amplitude for an extended nucleus. Now if $f(\theta)$ is to give a cross-section which agrees with experiment, it follows that at the angles of interest $f(\theta) \gg f'(\theta)$ so that

$$f(\theta) \approx -\frac{1}{2ik}(1-\mathrm{e}^{-2\varkappa})\sum_{j=\frac{1}{2}}^{j=j_{\max}} \mathrm{e}^{2i\lambda_j}(j+\tfrac{1}{2})\{P_{j-\frac{1}{2}}(\cos\theta)+P_{j+\frac{1}{2}}(\cos\theta)\}. \tag{4}$$

In general the $\varkappa$ and λ_j are not independent. However, if we calculate $\varkappa$ by introducing an imaginary term into the potential and using the Born approximation, then, for sufficiently high energies, we may use the W.K.B. method for a uniform charge distribution to give the λ_j, taking the imaginary part of the potential to be zero. This involves setting an upper limit to $\varkappa$ which we have taken to be 0.5 radians. The remaining conditions on $j_{\max}$ and $\varkappa$ are that $\varkappa > 0$ and that the cross-section for inelastic scattering of μ-mesons,

$$\sigma_{\text{inel}} = \frac{2\pi}{k^2}\sum_{j=\frac{1}{2}}^{j=j_{\max}} (j+\tfrac{1}{2})^2\left\{\frac{1}{2j}+\frac{1}{2j+2}\right\}\mathrm{e}^{-2\varkappa}\sinh 2\varkappa,$$

agrees with experiment.

If we suppose that the main inelastic process is the photo-production of π-mesons by the electromagnetic field of the incident particle, then, as a first guess, we may take the range of the imaginary part of the potential to be of the order of λ_{π_0}, the π-meson Compton wave-length, and this leads to $j_{\max} \approx 15/2$ for an incident energy of 1 GeV. With this value of $j_{\max}$ one quickly finds that the inelastic cross-section is far too large if $\varkappa = 0.5$ and, for a value of $\varkappa$ which approximately fits the π-meson production cross-section, the anomalous scattering cross-section is far too small.

However, Watson [7]) has shown that π-mesons of not too high energy are

produced by photons at an average distance of $\frac{1}{2}\lambda_{\pi c}$ from the centre of the nucleon. This leads to $j_{\max} \approx 7/2$. Actually we need not suppose that the inelastic are all due to π-meson production provided that the energy transfers involved are large enough to give a short range imaginary term in the potential.

Calculations of the projected angle distribution $F(\phi)$ have been made by expressing $f(\theta)$ in terms of a sum of Gauss functions and making the small angle approximation. At the angles of interest this turns out to be practically the same as the single scattering distribution. It is found that, assuming $\varkappa = 0.5$ and $\sigma_{\text{inel}} = 4$ mb for a lead nucleus, one gets fair agreement with experiment for a μ-meson energy of 1 GeV by taking $j_{\max} = 3/2$ and with the data for a μ-meson of 2 GeV by taking $j_{\max} = 9/2$. These results are shown in the figure together with the result for Z protons scattering independently (an upper limit to the ordinary Coulomb inelastic cross-section). Also shown is the differential cross-section calculated from

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\phi} = \frac{\pi\phi}{2}\left(\frac{2Ze^2E}{c(p\phi)^2}\right)^2,$$

where E and p are the μ-meson energy and momentum; this gives a rough representation of the experimental results. It should be added that at $p\phi \leqq 10$ the contribution form $f'(\theta)$ should be of the same order of magnitude as that from the second term of (3) and should therefore increase the theoretical cross-section for those values of $p\phi$.

For energies > 2 GeV the agreement is expected to be at least as good as that shown in the figure, from the dependence of σ_{inel} and $f(\theta)$ on k^2 and $j_{\max}$. For energies < 1 GeV, $j_{\max} = \frac{1}{2}$ and $\varkappa < 0.5$ so that the anomaly should become small, even if σ_{inel} does not decrease.

3. Discussion and Comparison with Experiment

Before proceeding with the comparison of theory and experiment, it is necessary to discuss in more detail the range of the imaginary potential referred to in § 2. One difficulty which arises is that the interpretation of the inelastic events in terms of the Williams equivalent photon method implies that $\frac{1}{2}\lambda_{\pi c}$ would be a lower limit to the range since it gives the dimension of the perturbed system. An upper limit to the range is then given by the adiabatic limit

$$b = \frac{\lambda_{\pi c}}{2} \cdot \frac{E}{m_\mu c^2},$$

which is of the order of the nuclear radius. This, however, neglects nucleon polarisation effects which should, by the analogy with the atomic case, introduce high energy Čerenkov radiation. The interpretation of these

phenomena in terms of an effective range of interaction then becomes rather obscure.

When more information becomes available one should be able to find

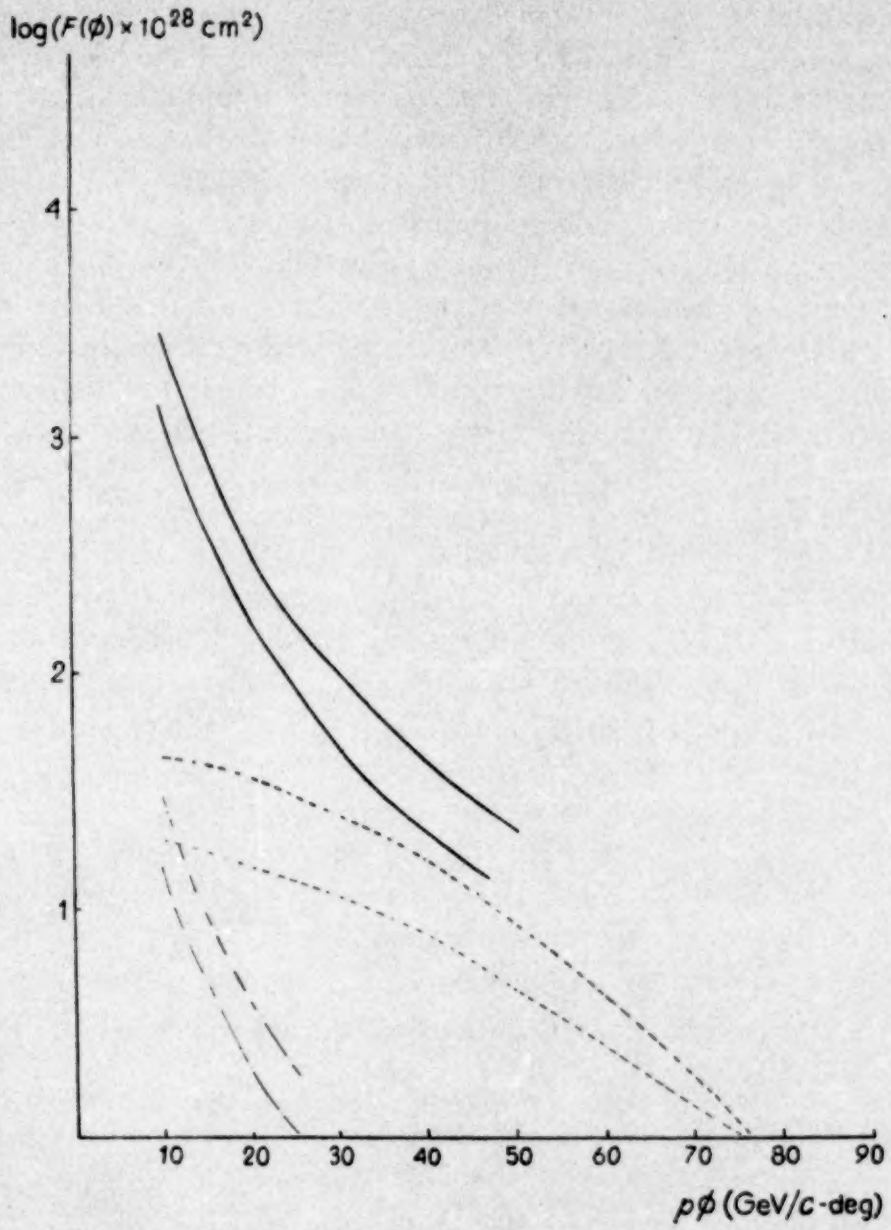

Fig. 1.

——— Approximate experimental results.
– – – – – $F(\phi)$ theoretical prediction of present work.
—·—· Upper limit to inelastic Coulomb scattering.
In each case the upper curve refers to an incident μ-meson of energy 2 GeV, the lower to an incident μ-meson of energy 1 GeV.

an independent estimate of j_{max} from the angular dependence of the inelastic scattering of μ-mesons involving energy transfers > 150 MeV.

As far as the comparison with experiment is concerned, it will be seen

from the figure that for $p\phi > 10$ the scattering produced by the effect described here approaches much more closely the scattering from a point nucleus than that produced by effects hitherto considered, and in fact the experimental results for the anomalous scattering cross-section so far given are not inconsistent with the values given here. However, one point should be noted, namely that the value which has been adopted for σ_{inel} is rather high. In fact for a mean μ-meson energy of ≈ 4 GeV, George has found a cross-section for production of stars of $\approx 5\times 10^{-30}$ cm^2/nucleon, or 1 mb for a lead nucleus. However, these were events which had at least one heavy prong and in which the incident and outgoing μ-meson could be reasonably distinguished from the appearance of the tracks. It is clear therefore that this result is certainly a lower limit and that the true inelastic cross-section may be considerably greater. For energies below 4 GeV there is no experimental information and theoretical predictions based on the Williams Weizsäcker method are certainly smaller than 1 mb for these energies. It must be supposed that processes not adequately treated by the Williams method must be present. Until controlled experiments are possible using the 1 GeV Stanford electron beam it seems that nothing further can be said for μ-mesons of these energies. For energies > 10 GeV, however, the star and shower production cross-section rises fairly rapidly [8]) and then 4 mb is certainly an under-estimate. (For energies < 1 GeV the inelastic cross-section should fall so that the anomaly should become much smaller.)

The other experimental data which may be compared with the theory are the Z or A dependence and the dependence on the sign of the charge of the μ-meson. Experiment [2]) gives a scattering proportional to Z^2 or A^2 and a slight excess of μ^+ to μ^- scattering at energies below about 5 GeV. The present theory predicts results proportional to A^2 from the $\varkappa$ dependence of eqs. (4) and (5) together with the fact that σ_{inel} should be proportional to A. No difference at these energies is predicted between the scattering of μ^+ and μ^-, and in fact the experimental evidence for this is based on a rather small number of events.

With regard to the energy-dependence of the effects, the data are summarised in ref. [9]): one finds that below 600 MeV the anomaly on scattering is certainly very small and that for greater energies the effects are significant. In addition some recent Russian measurements [10]) show no anomaly at energies below 150 MeV in agreement with expectation.

Thanks are due to Professor L. Rosenfeld for many interesting discussions and for drawing my attention to the recent Russian work on the subject, and also to Professor G. D. Rochester and Dr. A. W. Wolfendale and their collaborators for their constant interest and encouragement.

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ON THE CONSERVATION LAWS FOR WEAK INTERACTIONS

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Abstract: A variant of the theory is proposed in which non-conservation of parity can be introduced without assuming asymmetry of space with respect to inversion.

Various possible consequences of non-conservation of parity are considered which pertain to the properties of the neutrino and in this connection some processes involving neutrinos are examined on the assumption that the neutrino mass is exactly zero.

1. Combined Parity

As is well known, the unusual properties of K-mesons have created a perplexing situation in modern physics. The correlation between π-mesons in τ-decay ($K^+ \to 2\pi^+ + \pi^-$) leads to the necessity of assigning a 0^- state to K^+-mesons. This kind of system, however, cannot decay into two π-mesons ($K^+ \to \pi^+ + \pi^0$). We are thus faced with the dilemma of either assuming that two different K-mesons exist or that the conservation laws are violated in K-meson decay. In the first case one must then explain the identity of masses (which are equal to within two electron masses)and the near coincidence in lifetime of the θ and τ-decays. One may attempt to explain the equality of K-meson masses by postulating, as Lee and Yang [1]) have done, the existence of some hitherto unknown symmetry property of nuclear forces which transforms the τ-meson into a θ-meson. If, however, decay involving a neutrino ($K^+ \to \mu^+ + \nu$, $K^+ \to \mu^+ + \nu + \pi^0$, $K^+ \to e^+ + \pi^0 + \nu$) is considered to be essentially the same for particles of various parity a difference in lifetime related to the different rate of τ and θ-decay ($\approx 8\,\%$ and $\approx 25\,\%$) should be anticipated. This discrepancy should be not less than 30—40 %, a result which seems to be inconsistent with experiment[2]).

Thus we come to the conclusion that the hypothesis of the existence of two different K^+-mesons is contrary to the experimental facts and the only alternative is to assume that the generally accepted conservation laws are violated in K-decay. Since there is no reason to think that the law of conservation of angular momentum is untenable, we are apparently dealing here with a direct violation of the law of conservation of parity.

It might seem at first glance that non-conservation of parity implies asymmetry of space with respect to inversion. If, however, complete isotropy of space (conservation of angular momentum) is taken into account this

type of asymmetry would seem to be extremely strange and in my opinion a simple rejection of parity conservation would create a difficult situation in theoretical physics. I would like to point out a solution of this problem which consists in the following. As is well known, both the law of conservation of parity and charge conjugation invariance undoubtedly hold in strong interactions. Let us now assume that each of these conservation laws does not hold separately in weak interactions. However, invariance with respect to the set of both operations (which we shall call combined inversion) will be assumed to exist. In combined inversion, space inversion and transformation of a particle into an antiparticle occur simultaneously.

It is easy to see that invariance of the interactions with respect to combined inversion leaves space completely symmetrical, and only the electrical charges will be asymmetrical. The effect of this asymmetry on the symmetry of space is no greater than that due to chemical stereo-isomerism.

On the other hand the law of conservation of parity of charged particles will not hold as the operator of combined inversion does not transform charged particles into themselves.

Furthermore, it is easy to see that the constants characterizing the particles and antiparticles (masses, lifetimes) should be identical since, as a result of invariance with respect to combined inversion, all processes involving particles and antiparticles should differ from each other only in regard to space inversion. Graphically speaking, a K^--meson is a mirror reflected K^+-meson.

Truly neutral particles, that is, particles which are identical to their antiparticles, transform into themselves in combined inversion. Consequently, with respect to these particles combined inversion leads to a law of conservation of combined parity. It should be emphasized that conservable parity is the product of ordinary parity and charge parity of the particles. Evidently, in this sense the π^0-meson is an odd particle; the $K_1^0(\theta^0)$-meson which decays into 2 π-mesons is an even particle and the K_2^0-meson predicted by Gell-Mann and Pais [3]) and recently discovered experimentally [4]) is an odd particle. Combined inversion changes the sign of the magnetic field of a photon but does not change that of the electric field. The ordinary parities of electric and magnetic multipoles are reversed for combined inversion.

It is easy to show from the foregoing that despite the absence of ordinary parity the particles cannot possess dipole moments. Indeed, the only vector which can be constructed from ψ-operators for a particle at rest is its spin vector which is even with respect to inversion and odd with respect to charge. It is consequently odd with respect to combined inversion and, in accord with the foregoing regarding the electromagnetic field, it defines only a magnetic but not an electric moment.

Lee and Yang [5]) † have shown that non-conservation of parity leads to correlations in a number of hyperon production and decay processes. It can be shown that a consequence of invariance with respect to combined inversion is that the weak interaction operators in the Lagrangian contain real coefficients. This circumstance, however, does not appreciably modify the qualitative picture which is obtained in the general case of non-conservation of parity. Therefore asymmetry of hyperon decay with respect to the plane of their creation, which has been predicted by Lee and Yang [5]), will also hold in this case.

I would like to express my deep appreciation to L. Okun, B. Ioffe and A. Rudik for discussions from which the idea of this part of the present paper emerged.

2. Properties of the Neutrino

Rejection of the law of conservation of parity entails the possibility of existence of new properties of the neutrino. The Dirac equation for the case of zero mass splits into two independent pairs of equations. It will be recalled that in the usual theory one cannot confine oneself to a single pair of equations since both pairs transform into each other as a result of space inversion. If, however, we restrict our attention to combined inversion we arrive at the possibility of describing the neutrino by a single pair of equations. In the sense of the usual scheme this would signify that the neutrino is always polarized in the direction of its motion (or in the opposite direction). The polarization of the antineutrino is correspondingly reversed. According to this model the neutrino is not a truly neutral particle and this agrees with the fact that double β-decay has not been observed experimentally and especially with the results of experiments on induced β-decay. We shall call this kind of neutrino a longitudinally polarized neutrino or briefly a longitudinal neutrino.

In the usual theory the neutrino mass is zero, so to say, accidentally. Thus, account of neutrino interactions automatically leads to the appearance of a definite, albeit vanishingly small, rest mass. The mass of the longitudinal neutrino, on the other hand, vanishes automatically and this situation cannot be altered by the existence of any type of interaction.

The longitudinal neutrino concept appreciably reduces the possible number of types of weak interaction operators. Consider, for example, the decay of a μ-meson into an electron and two neutrinos. In the usual manner we represent the interaction operator as the product of operators consisting of μ-meson and electron ψ-operators on the one hand and ψ-operators of the two neutrinos on the other. For the longitudinal neutrino only one combination can be made from the two neutrino operators—a scalar (a

† I would like to sincerely thank the authors for sending me a preprint of their paper.

scalar with respect to rotation; the operation of ordinary inversion is not applicable), as it is well known that the tensor combination of two identical operators obeying Fermi statistics is equal to zero. In this case two combinations, scalar and pseudo-scalar (in the usual sense of the word), can be constructed for a μ-meson and electron.

If a neutrino and antineutrino are emitted in μ-meson decay the situation changes. Only a four-dimensional vector can then be constructed from the longitudinal neutrino and antineutrino operators. In this case two combinations — vector and pseudo-vector—can be made from the μ-meson and electron operators. Thus, despite the absence of invariance with respect to inversion in each of the two cases only two interaction operators are possible.

It is easy to calculate the energy spectrum of the μ-meson decay electrons. It is found to be exactly the same as that calculated by Michel [6]). The two-neutrino case thus yields for Michel's constant, ϱ, the value $\varrho = 0$ and for a neutrino and antineutrino $\varrho = 0.75$. The former case is apparently inconsistent with the experiments, whereas the latter agrees with the results obtained in ref. [7, 8]), which yield $\varrho = 0.64 \pm 0.10$ and $\varrho = 0.57 \pm 0.14$. Thus μ-meson disintegration experiments do not contradict the longitudinal neutrino concept and in this case lead to a unique result, namely, that a neutrino and antineutrino are involved in μ-meson decay.

Consider now the reaction $\pi \to \mu + \nu$. Since the π-meson is spinless we are obliged to set up a scalar expression for the μ-meson and neutrino ν-operators in the $\pi \to \mu + \nu$ decay operator. This automatically yields that if the neutrino is longitudinal the μ-mesons produced in $\pi \to \mu + \nu$ decays will be completely polarized in the direction of their motion (or in the opposite direction).

As Lee and Yang [5]) have noted, a possible consequence of non-conservation of parity is the correlation between the directions of the μ-meson and electron involved in the $\pi \to \mu \to$ e decay. Simple calculations based on our scheme give the following energy and angular distribution for the emitted electrons:

$$\frac{dN}{N} = 2\varepsilon^2[(3-2\varepsilon)+\lambda \cos\vartheta(2\varepsilon-1)]d\varepsilon. \qquad (1)$$

Here ε is the ratio of electron energy to the largest possible energy, ϑ is the angle between the directions of motion of the μ-meson and electron and λ is a constant which depends on the relation between the vector and pseudovector parts in the combination of the μ-meson and electron ψ-operators,

$$\lambda = \frac{2ab}{a^2+b^2}, \qquad (2)$$

where a and b are coefficients of the respective terms and, according to the foregoing, are real. Evidently, λ varies between -1 and $+1$. It is possible that λ is in fact equal to zero. The integral electron distribution is obviously proportional to $(1+\frac{1}{3}\lambda\cos\vartheta)$ and this means that the largest possible value of the forward-backward asymmetry is 2. It should be noted that even if λ appreciably differs from zero it may be difficult to observe $\mu \to$ e correlation because of depolarization of the slowed down mesons and in particular for μ^+-mesons because of formation of mesonium ($\mu^+ + e^-$ system).

Consider now the effect of longitudinality of the neutrino on β-decay. According to experiment the decay operator should be represented as the sum of the scalar and tensor variants. It can be shown that in either case the same electron polarization in the direction of motion will arise, which is equal to v/c (or $-v/c$), the ratio of the electron velocity to that of light. Thus high energy electrons will be totally polarized in the direction of their motion.

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AN ANTI-REVIEW

[EDITOR'S NOTE: Arnold Berliner, the Editor of *Naturwissenschaften*, used to say that authors react to criticism as if they were hybrids of hedgehog and mimosa. The publication of authors' protests against unappreciative reviews of their works must, however, to avoid monotony, remain restricted to exceptional cases, judged on their own merit. Prof. Landé, in the following rejoinder to our review of his recent book, does not merely give vent to his wrath at the reviewer's inability to see the point of trying to banish the quantum from quantum theory, and advertise an article by Mario Bunge as an 'antidote' against the reviewer's own attitude to this theory; but he submits a re-statement of his six postulates, "more clearly formulated now than in the book written three years ago". While regretting that the reviewer's 'selfassurance' should be such a cause of irritation to his eminent colleague, the Editor is glad to note that the review had also been an occasion for him to present his views in what he regards as an improved form. There was therefore no hesitation on the Editor's part in deciding to publish Prof. Landé's comments. Whether they will help his cause is another question, which may safely be left to the judgment of the readers — doctored or not with Bunge's Antidote.]

COMMENTS ON THE REVIEW OF "FOUNDATIONS OF QUANTUM MECHANICS, A STUDY IN CONTINUITY AND SYMMETRY"

ALFRED LANDÉ

Ohio State University, Columbus, Ohio

The Book Review by L. R. in this Journal [**1** (1956) 133] contains too many misleading statements to remain unanswered. Here is the author's reply. L.R.'s sharpest criticism is directed against the *intent* of the book which "chooses to invoke, of all people, Leibniz, and invites us to be guided by the creed that the world is not only the best but also the simplest of all possible worlds". This paraphrase of Leibniz is an ornamental flourish decorating the last section of the last chapter of the book, and it can hardly be called an "invitation" except by those who read backwards. But just the same, what does L. R. want? The *search* for the simplest possible laws has occupied scientists from Thales via Leibniz to modern times. Yet the *creed* that there *are* simple laws to discover, that "things run their course in an intelligible way", is scored as "the plague of our times", and a book with such quotations from Leibniz must be condemned as "metaphysics". As to the more specific accusation that the author "would reduce the infinite richness of atomic laws (read phenomena?) to the shadowy Leibnizian principles of continuity and symmetry", L. R. overestimates the power of the author who unfortunately needed as many as *six* shadowy postulates, rather than the two which the reviewer found on the book's cover. The six are compiled at the end of this reply. L. R. thinks that this "venture into thin air . . . makes a muddle out of a perfectly clear situation". Which situation? L. R. probably refers to duality and uncertainty as illuminated by Bohr's complementarity of conjugate observables more than 25 years ago. This clarifying idea has since been expounded in an unending flood of interpretation and commentary. But all this philosophical talk about complementarity avoids to attack the physical problem which would create a "perfectly clear situation", namely, that of *why*, i.e. on what elementary grounds, conjugate observables are in a periodic hence complementary relation. To suggest that this problem exists, that the theory might be reduced from duality and complementarity further to more rational, simple, common-sense ground postulates may seem naive, or subversive, or both to L. R. To this author and to an increasing number of others it appears quite otherwise, namely, that the meta physical belief in com-

plementarity as the last and ultimate and irreducible principle of microphysics is a dogmatic stumbling block, traceable to a positivistic philosophy which insists that description (and dialectic circumscription) is all we need, and that explanation is pointless. The positivist view of renunciation to explain has succeeded in securing a stronghold among physicists chiefly because it is so very *convenient*, when one does not find a solution, to declare the problem as insoluble "in principle", or to deny that the problem exists. And indeed, those earlier attempts at explaining Planck's and Bohr's quantum rules completely failed, since an inconsistent structure cannot be reduced to a consistent one. With the advent of the self-consistent theory of 1926 the situation has radically changed, and to heed the earlier lesson of irreducibility has become an anachronism. Let the future decide whether the author's positive attempt to deduce quantum mechanics from non-quantal postulates is acceptable at least as a first step or whether he has "made a muddle of a perfectly clear situation". But let us not a priori condemn all attempts to go beyond complementarity because L. R. so decrees.

Turning now to the more technical *content* of the book, L. R. complains that it "suggests a connection between the statistical aspects of the quantum theory and thermodynamics. There is of course no such connection". Why the categorical "of course"? The book points out that the concept of 'total inequality' or total separability of two different *kinds* of particles (with a corresponding entropy of diffusion of their gases) may, in application to *states* of the same kind of particles, be generalized to the concept of a 'fractional equality' (corresponding to a less than classical diffusion entropy). This 'fractional equality' must operationally be interpreted as the transition probability between the states in a test, as set forth in the book. An important quantum feature can thus be developed from the non-quantal postulate of continuity (applied to the diffusion entropy). And when Gibbs' entropy discontinuity, occurring when two different *kinds* of particles are gradually changed into the same kind, is overcome by the very existence of atoms and molecules (as L. R. correctly remarks), then the Gibbs paradox for different *states* is overcome by 'fractional equality degrees' manifested as transitions in tests (which L. R. seems unable to grasp). All this might indeed be developed without mentioning thermodynamic implications such as the Gibbs paradox. But to "suggest" that such connections exist does not do any harm. On the contrary, one of our leading theorists renowned for his critical vein displayed a rather sceptical attitude toward the author's "venture into thin air" until he found the filters used in the thought-experiments of the book to be legitimate generalizations of the semipermeable diaphragms used in thought-experiments of thermodynamics. L. R. further protests that in case of identical particles "the removal of the $N!$ fold degeneracy has nothing to do with the removal of the Gibbs paradox, as the author contends". He may change his mind when he reads Schrödinger's classic monography "Statistical Thermodynamics".

Altogether, L. R.'s "nothing to do's", his "of course's", his "plague" his "thin air" and "muddle" and so forth seem less inspired by a conscientious effort of the reviewer to understand and to report a new approach to the fundamental problem of periodicity in quantum mechanics than by an apparent desire to defend duality and complementarity as *ultimate* principles at all costs. L. R. gives himself away in this respect by his concluding remark: "*The real* (italics mine) introduction to the foundations of quantum theory exists. It is Niels Bohr's Discussion with Einstein . . ." With due respect for a faithful disciple, does L. R. actually believe that *the real* wisdom is to be found only in Copenhagen and surrounding feodalities? Did his self-assurance not suffer from the detailed analysis by a prominent physicist-philosopher who found that the 'dialectics' produced in L. R.'s own 'Strife about Complementarity' is full of holes, obviously false claims, and abstrusities? We refer to Mario Bunge's counter article "Strife about Complementarity" which appeared in Brit. J. Philosophy of Science [**6** (1955) 1 and 141] and which every physicist ought to read as an antidote.

Returning to the author's own misbegotten book: In spite of Quotations from Leibniz it did not have philosophical aspirations at all but tried to make a contribution to a problem of *theoretical physics*, namely, that of reducing the strange concepts and rules of quantum mechanics, including complementarity, to more elementary postulates. In order to give the readers of this Journal a more authentic report, here are the six postulates, more clearly formulated now than in the book written three years ago:

1) The continuity postulate; it leads to the admission of 'fractional equality' between states
2) Reproducibility of a measurement; combination of (1) and (2) leads to transitions between states in 'filter' tests, controlled by statistical rules.
3) Symmetry of the transition probability, $P(A, B) = P(B, A)$, in correspondence to classical reversibility.
4) The Second Law of thermodynamics; (3) and (4) together lead to 'magic square' matrices of the probabilities, with totally separable states figuring as 'mutually orthogonal'. states.
5) The postulate that there is a general metric law connecting the probability matrices; the only possibility to construct such a general law is that of orthogonal transformation, which is the superposition law for complex amplitudes Ψ.
6) A constant probability density in qp-space; (5) and (6) together necessitate a periodic function $\Psi(q, p)$ of the complex-imaginary form exp $(2i\pi qp/\text{const})$, in which the constant may be denoted by the now familiar letter h. The periodic qp-relation (duality) and other quantum features are thus deducible from, i.e. explained on, a non-quantal basis.

Postscript. The author fails to see what is to be gained by the incorrect remark in the Editor's note, above, that he is "trying to banish the quantum from quantum theory", when in fact he is trying to show why there is an action constant h connecting q and p in a periodic fashion.

BOOK REVIEW

M. H. SHAMOS and G. M. MURPHY (editors), *Recent advances in science — physics and applied mathematics* (New York University Press — Interscience Publishers, New York 1956. xi-384 pages. $ 7.50)

In 1954 the Division of General Education of New York University organised a symposium on Recent Advances in Science for the benetit of specialised scientists curious of learning about the kind of problems with which their colleagues are grappling. The lectures, all by eminent authorities in their respectıve fields, now appear in book-form: they have lost nothing of their value and will enable the great service which a symposium like this renders to the cause of scientific culture to spread to a world audience

The choice of subjects did not claim to embrace all the fields in which active research is going on; there is nothing for instance, on modern optics or on such perhaps too esoteric problems as those of theoretical thermodynamics. On the other hand, one may well wonder why the topic 'atomic structure' was considered at all (the lecturer made an excellent job of it, however, in spite of the fact that he had no startling news to report). Nuclear physicists will not be surprised, of course, to find that their doings are allotted the lion's share, about half of the book, in fact. It is worth while reproducing the table of contents, to give an idea of the selection and general set-up:

Methods of applied mathematics (R. Courant)
The future of operational research (P. M. Morse)
Atomic structure (J. J. Rabi)
Microwave spectroscopy (C. H. Townes)
Nuclear structure and transmutations (H. A. Bethe)
Elementary particles (V. F. Weisskopf)
Electronuclear machines (L. J. Haworth)
Neutron physics (N. F. Ramsey)
Transistor physics (W. Shockley)
Ferromagnetism (R. M. Bozorth)
Cryogenics: very-low temperature physics and engineering (F. G. Brickwedde)
Physics and the engineer (E. U. Condon).

The style of the lectures has fortunately been preserved in most cases: the account is broad, fairly elementary, or at least free from too heavy technicalities. The lecturers have been at pains to get across the essentials, and (being the people they are) they have done so very successfully, brilliantly even. Some of the articles, which seem too long for a normal lecture, have presumably been extended for the purpose of publication; but this has not spoiled in the least their character of introductory surveys. Most contributions are excellently illustrated, not only by well chosen photographs, but above all by clear and ingeniously conceived explanatory diagrams. This is a good contribution to a good cause. May it not remain isolated.

L. R.

NEWS ITEM

SIXTH SESSION OF THE COUNCIL OF CERN (GENEVA, 14 DECEMBER 1956)

Among a number of measures approved at the Sixth Session of the CERN Council, which was held in Geneva on 14th December, 1956, the most important were the estimates for the financial year 1957, the capital investment programme for the 1952—1960 construction period, the decision taken concerning CERN's Patent Policy and, finally, the election of the Committee of Council.

The estimated expenditure for 1957 was established and accepted at 61800000 Swiss Francs. This compares with a figure of 39600000 Swiss Francs for 1956. The difference reflects the fact that CERN's construction activity at Meyrin is entering more and more into a period of peak development, what with the removal of personnel to the site, the civil engineering involved in the construction of the machines themselves, and the implementation of the scientific research programme. This latter has become the more urgent inasmuch as the first of CERN's two accelerators, the Synchrocyclotron, will be in operation by the second half of 1957.

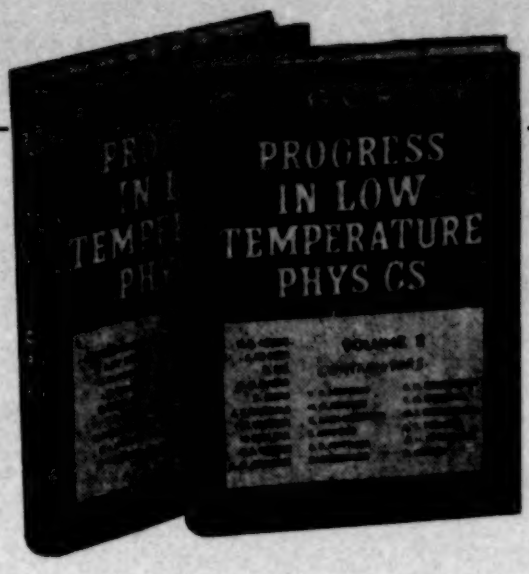
With regard to the long-term capital investment programme, i.e. that for the actual construction period from 1952 to 1960, the Council approved a figure of 219000000 Swiss Francs Cost of construction is estimated at 166000000 Swiss Francs, and operating expenses — covering research, theory, experimentation and maintenance — at an amount of 53000000 Swiss Francs. Thus the increased estimates reflected chiefly the development of research, the development of ancillary and common services and, lastly, the extension and increased cost of construction.

It is expected that staff will increase from 420 in 1956 to 513 in 1957 (about 22 %) in direct CERN staff, and from 449 to 558 (about 24 %) when Fellows and Guest Professors are included. Finally, in 1960, it is foreseen that the total CERN staff, including Fellows and Guest Professors, will amount to about 600.

Patent Policy, as previously mentioned, was also an item discussed at the CERN Council Session. It was felt that in view of CERN's character as an international body set up for the purpose of pure research, an active patent policy with its consequent trouble and expense, accompanied by the possibility of a disturbance in morale and a distortion of the research programme would not be indicated, and that it would be preferable to provide merely for the simple publication of any inventions produced. The Council, therefore, decided that as a general rule no patent protection will be sought for any inventions produced in its laboratories, but that these will be published to the world at large so as to make their use freely available to science and industry everywhere. The Council however authorized the Director-General in very exceptional cases of major economic importance to take such measures, including patent action, as may seem appropriate.

Finally, elections held at the end of the Session resulted in appointments to the Committee of the Council, the central of its three subordinate committees:

President: Sir Ben Lockspeiser, as President of the Council (United Kingdom), re-elected	
Vice-Presidents of the Council:	M. F. de Rose (France) Professor J. Holtsmark (Norway)
Chairman of the Finance Committee:	M. J. Willems (Belgium), re-elected
Chairman of the Scientific Policy Committee:	Professor W. Heisenberg (German Federal Republic)
Additional Members of the Committee of Council	Dr. A. Pennetta (Italy) Dr. I. Supek (Yugoslavia).



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